CONVERGENCE OF ORTHOGONAL RATIONAL FUNCTIONS

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1 Introduction

Every finite positive Borel measure μ on the unit circle gives rise to an orthonormal sequence $\{\varphi_n\}$ of polynomials: Szegő polynomials. The reciprocal polynomials $\{\varphi_n^*\}$ are defined by $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$. The (positive) leading cofficient of $\varphi_n(z)$ is called κ_n, it is easily seen that $\kappa_n = \varphi_n^*(0)$. The reflection coefficient δ_n is defined by $\delta_n = \varphi_n(0)/\kappa_n$. The Szegő polynomials and their reciprocals satisfy the coupled recurrence relations

$$\varphi_n(z) = \frac{\kappa_n}{\kappa_{n-1}}[z\,\varphi_{n-1}(z) + \delta_n\,\varphi_{n-1}^*(z)] \tag{1.1}$$

$$\varphi_n^*(z) = \frac{\kappa_n}{\kappa_{n-1}}[\overline{\delta_n}z\varphi_{n-1}(z) + \varphi_{n-1}^*(z)] \tag{1.2}$$

$$\varphi_0 = \varphi_0^* = \kappa_0. \tag{1.3}$$

The measure μ satisfies the Szegő condition $\int_{-\pi}^{\pi} \ln \mu'(\theta)d\theta > -\infty$ if and only if $\{\kappa_n\}$ converges to a finite value κ and if and only if the series $\sum_{n=1}^{\infty} |\delta_n|^2$ converges. When these equivalent conditions are satisfied, the sequence $\{\varphi_n^*\}$ converges locally uniformly in the open unit disk to the function

$$\pi(z) = \frac{1}{\sqrt{2\pi}}\exp[-\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\,\mu'(\theta)\,d\theta]. \tag{1.4}$$

When the stronger condition $\sum_{n=1}^{\infty} |\delta_n| < \infty$ is satisfied, the sequence $\{\varphi_n^*\}$ converges uniformly on the closed unit disk, and the (continuous) limit function does not vanish. For proofs and more details concerning these matters, see [12,13,14,15,16,18,23].

Polynomials are rational functions with a pole at infinity only. For some purposes it is useful to work with orthogonal rational functions with prescribed poles outside the unit disk. From a purely mathematical point of view the theory of such orthogonal rational functions was as far as we know initiated by Djrbashian about 1960 (see the survey paper [11]). Independently, partly from an applied point of view, the same constructions were used by Bultheel, Bultheel and Dewilde, Dewilde and Dym about 1980 (see [1,2,10]). The basic features of a general theory of orthogonal rational functions are set forth in [3,4,5,6,7,8] and in references found there. See also [20,21]. The theory is closely connected with the Nevanlinna-Pick interpolation problem. See e.g. [19, 20, 21], and for applications [9].

In [8] the complex of conditions related to the Szegö condition in the rational situation was investigated under the assumption that all the poles are contained in a compact subset (in the extended complex plane $\hat{C}$) of the exterior of the unit disk. It was established that (when the Szegö condition is satisfied) subsequences of the reciprocals $\{\varphi_n^*\}$ of the orthogonal rational functions $\{\varphi_n\}$ converge locally uniformly in the open unit disk to functions

$$\pi_\alpha(z) = e^{-i\lambda}\frac{\sqrt{1-|\alpha|^2}}{\sqrt{2\pi}(1-\overline{\alpha}z)}\exp[-\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\mu'(\theta)\,d\theta],\ \ \lambda\in R.$$

In particular the whole sequence $\{\varphi_n^*\}$ converges to π_α if the prescribed poles converge to $1/\overline{\alpha}$.

The main result of this paper is a theorem on uniform convergence of $\{\varphi_n^*\}$ on the closed unit disk which generalizes the theorem referred to above for the polynomial case.

2 Orthogonal functions

We shal use the notations $T = \{z \in C : |z| = 1\}$, $D = \{z \in C : |z| < 1\}$, $E = \{z \in C : |z| > 1\}$. The substar conjugate f_* of a function f is defined by

$$f_*(z) = \overline{f(1/\overline{z})}. \tag{2.1}$$

Let μ be a finite (positive) Borel measure on $(-\pi,\pi]$. An inner product is defined by

$$<f,g> = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\mu(\theta) = \int_{-\pi}^{\pi} f(e^{i\theta})g_*(e^{i\theta})d\mu(\theta). \tag{2.2}$$

Let $\{\alpha_n\}$ be an arbitrary sequence of (not necessarily distinct) points in D. It is sometimes convenient to use the notation $\alpha_0 = 0$. The Blaschke factors ζ_n are defined by

$$\zeta_n(z) = \tau_n\frac{(\alpha_n - z)}{(1-\overline{\alpha_n}z)},\ \text{ where } \tau_n = \frac{\overline{\alpha_n}}{|\alpha_n|},\ n = 1,2,\dots. \tag{2.3}$$

(By convention, $\tau_n = -1$ when $\alpha_n = 0$.) The Blaschke products B_n are defined by

$$B_0(z) = 1, \quad B_n(z) = \prod_{k=1}^{n} \zeta_k(z) \text{ for } n = 1, 2, \ldots. \tag{2.4}$$

We define the spaces $\mathcal{L}_n$ by

$$\mathcal{L}_n = Span\,\{B_m : m = 0, 1, ..., n\}. \tag{2.5}$$

The elements of $\mathcal{L}_n$ are exactly the functions that may be written in the form

$$L(z) = \frac{p_n(z)}{\pi_n(z)} \tag{2.6}$$

where

$$\pi_n(z) = \prod_{k=1}^{n}(1 - \overline{\alpha_k} z) \tag{2.7}$$

and $p_n \in \Pi_n$ (the space of polynomials of degree at most n). In particular $\mathcal{L}_n = \prod_n$ when $\alpha_n = 0$ for all n.

For $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ we define the superstar conjugate f^* by

$$f^*(z) = B_n(z) f_*(z). \tag{2.8}$$

Observe that $f^* \in \mathcal{L}_n$.

Let the sequence $\{\Phi_n : n = 0, 1, 2, ...\}$ be obtained by orthogonalization of the sequence $\{B_n : n = 0, 1, 2, ...\}$ with respect to the inner product (2.2). Each Φ_n has a decomposition

$$\Phi_n(z) = \sum_{k=0}^{n} b_k^{(n)} B_k(z). \tag{2.9}$$

By calculating $\Phi_n^*(z)$ and substituting α_n for z we see that

$$\overline{\Phi_n^*(\alpha_n)} = b_n^{(n)}. \tag{2.10}$$

We shall reserve the notation Φ_n for the monic functions, i.e. those for which $b_n^{(n)} = 1$. We set

$$\kappa_n = < \Phi_n, \Phi_n >^{-\frac{1}{2}} \tag{2.11}$$

and denote by φ_n the normalized functions:

$$\varphi_n(z) = \kappa_n \Phi_n(z). \tag{2.12}$$

We thus have

$$< \varphi_n, \varphi_n > = < \varphi_n^*, \varphi_n^* > = 1 \tag{2.13}$$

$$\varphi_n^*(\alpha_n) = \kappa_n, \;\; \Phi_n^*(\alpha_n) = 1. \tag{2.14}$$

It can be shown that φ_n^* solves the extremum problem

$$\max\{|f(\alpha_n)| : f \in \mathcal{L}_n, < f, f > = 1\}. \tag{2.15}$$

The sequence $\{\mathcal{L}_n\}$ is nested, i.e. $\mathcal{L}_n \subset \mathcal{L}_{n+1}$. It follows that the sequence $\{\kappa_n\} = \{\varphi_n^*(\alpha_n)\}$ is non-decreasing if $\alpha_n = \alpha$ for all n. This monotonicity property of $\{\kappa_n\}$ does not follow in general.

The functions φ_n have all their zeros in D, while the functions φ_n^* have all their zeros in E.

The sequences $\{\varphi_n\}$, $\{\varphi_n^*\}$ satisfy the following coupled recurrence relations:

$$\varphi_n(z) = \frac{\kappa_n}{\kappa_{n-1}} [\varepsilon_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \varphi_{n-1}(z) + \delta_n \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \varphi_{n-1}^*(z)] \quad n = 1, 2, ... \tag{2.16}$$

$$\varphi_n^*(z) = -\tau_n \frac{\kappa_n}{\kappa_{n-1}} [\overline{\delta_n} \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_n} z} \varphi_{n-1}(z) + \overline{\varepsilon_n} \frac{1 - \overline{\alpha_{n-1}} z}{1 - \overline{\alpha_n} z} \varphi_{n-1}^*(z)] \quad n = 1, 2, ... \tag{2.17}$$

$$\alpha_0 = 0, \;\; \varphi_0 = \kappa_0, \;\; \varphi_0^* = \kappa_0. \tag{2.18}$$

The recurrence coefficients δ_n, ε_n are given by

$$\delta_n = \frac{(1 - \alpha_{n-1}\overline{\alpha_n})}{(1 - |\alpha_{n-1}|^2)} \frac{\varphi_n(\alpha_{n-1})}{\kappa_n} \tag{2.19}$$

$$\varepsilon_n = -\tau_n \frac{(1 - \overline{\alpha_{n-1}}\alpha_n)\overline{\varphi_n^*(\alpha_{n-1})}}{(1 - |\alpha_{n-1}|^2)\kappa_n}. \tag{2.20}$$

The coefficient ε_n is always different from zero. (Follows e.g. by substituting $z = \alpha_{n-1}$ in (2.17), taking into account that φ_n^* has no zeros in D.) Furthermore $|\delta_n| < |\varepsilon_n|$.

The sequences $\{\kappa_n\}$, $\{\delta_n\}$, $\{\varepsilon_n\}$ are connected through the formula

$$\frac{\kappa_0^2}{\kappa_n^2}(1 - |\alpha_n|^2) = \prod_{m=1}^{n} [|\varepsilon_m|^2 - |\delta_m|^2]. \tag{2.21}$$

Clearly $\{\frac{\kappa_n}{\sqrt{1 - |\alpha_n|^2}}\}$ converges to a finite value if and only if the infinite product $\prod_{m=1}^{\infty} [|\varepsilon_m|^2 - |\delta_m|^2]$ converges (to a finite value different from zero).

For more exhaustive treatments of the concepts and results referred to, see [3,4,5,6,7,8].

3 Locally uniform convergence

We shall in the rest of this paper assume that the sequence $\{\alpha_n\}$ converges to a limit α in D:

$$\lim_{n\to\infty} \alpha_n = \alpha, \;\; \alpha \in D. \tag{3.1}$$

(More general results can be obtained by only assuming that all α_n are contained in a compact subset of D, by considering accumulation points and convergent subsequences. Cf. [8].)

We recall that the measure μ is said to satisfy the Szegő condition if

$$\int_{-\pi}^{\pi} \ln \mu'(\theta)\, d\theta > -\infty. \tag{3.2}$$

(Here $\mu'(\theta)$ denotes the derivative of μ - and of the absolutely continuous part of μ - with respect to Lebesgue measure.)

We now state as a theorem some basic results connecting the Szegő condition with behavior of the sequences $\{\kappa_n\}$, $\{\delta_n\}$, $\{\varepsilon_n\}$:

Theorem 3.1 *The following conditions are equivalent when (3.1) is satisfied:*

(A) $\{\kappa_n\}$ *converges (to a finite value different from zero).*

(B) $\prod_{m=1}^{\infty}[|\varepsilon_m|^2 - |\delta_m|^2]$ *converges (to a finite value different from zero).*

(C) *The Szegö condition (3.2) is satisfied.*

Proof: Follows from [8, Theorem 6.10].

□

The Szegö spectral factor σ_μ is defined by

$$\sigma_\mu(z) = \sqrt{2\pi}\exp[\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\ln\mu'(\theta)d\theta]. \tag{3.3}$$

(We may set $\sigma_\mu \equiv 0$ when the Szegö condition (3.2) is not satisfied.)

We state a main result on convergence of the sequence $\{\varphi_n^*\}$.

Theorem 3.2 *Assume that (3.1) and (3.2) (hence the conditions (A), (B), (C)) are satisfied. Then* $\{\varphi_n^*\}$ *converges locally uniformly in* D *to a function*

$$\pi_\alpha(z) = e^{-i\lambda}\frac{\sqrt{1-|\alpha|^2}}{(1-\overline{\alpha}z)\sigma_\mu(z)}, \ \lambda \in R. \tag{3.4}$$

Proof: Follows from [8, Theorem 6.10 and Theorem 6.12].

□

Corollary 3.3 *Assume that*

$$\alpha_n = \alpha \quad \text{for all } n \tag{3.5}$$

and that

$$\sum_{n=1}^{\infty}|\delta_n|^2 < \infty. \tag{3.6}$$

Then $\{\varphi_n^*\}$ *converges locally uniformly in* D *to the function* π_α *given by (3.4).*

Proof: It follows from (2.14), (2.20) and (3.5) that in this case $|\varepsilon_n| = 1$, and so (3.6) is equivalent to condition (B). The result then follows from Theorem 3.1 and Theorem 3.2.

□

For the sake of completeness we also include the following result.

Theorem 3.4 *Assume that (3.1) and (3.2) (hence the conditions (A), (B), (C)) are satisfied. Then the reproducing kernels*

$$k_n(z,w) = \sum_{m=0}^{n} \varphi_m(z)\,\overline{\varphi_m(w)} \tag{3.7}$$

converge locally uniformly for $z, w \in D$ to the function

$$s(z,w) = \frac{1}{(1 - z\overline{w})\sigma_\mu(z)\overline{\sigma_\mu(w)}}, \tag{3.8}$$

i.e.

$$\sum_{m=0}^{\infty} \varphi_m(z)\overline{\varphi_m(w)} = \frac{1}{(1 - z\overline{w})\sigma_\mu(z)\overline{\sigma_\mu(w)}}. \tag{3.9}$$

Proof: Follows from [8, Theorem 6.14].

□

4 Uniform convergence

We recall that we shall also in this section assume that (3.1) is satisfied. We shall prove stronger convergence results for $\{\varphi_n^*\}$ under a stronger condition than (3.2). This condition is

$$\sum_{m=1}^{\infty} [|1 + \tau_m \overline{\varepsilon_m}| + |\delta_m|] < \infty. \tag{4.1}$$

We shall first establish that this condition is in fact stronger than (3.2).

Proposition 4.1 *Assume that (4.1) is satisfied. Then the product*

$$\prod_{m=1}^{\infty} [|\varepsilon_m|^2 - |\delta_m|^2] \tag{4.2}$$

converges absolutely.

Proof: First note that

$$||\varepsilon_m| - 1| \leq |1 + \tau_m \overline{\varepsilon}_m| \tag{4.3}$$

and

$$||\varepsilon_m|^2 - 1| \leq ||\varepsilon_m| - 1|(|\varepsilon_m| + 1). \tag{4.4}$$

It follows from (4.1) that

$$|\varepsilon_m| + 1 \leq M < \infty, \ \sum_{m=1}^{\infty} |\delta_m|^2 < \infty. \tag{4.5}$$

Together (4.1) - (4.4) give

$$\sum_{m=1}^{\infty} |(|\varepsilon_m|^2 - 1) - |\delta_m|^2| < \infty. \tag{4.6}$$

From a basic convergence criterion for infinite products (see e.g. [17]) we then conclude that the product

$$\prod_{m=1}^{\infty} [|\varepsilon_m|^2 - |\delta_m|^2] = \prod_{m=1}^{\infty} [1 + (|\varepsilon_m|^2 - 1 - |\delta_m|^2)] \tag{4.7}$$

converges absolutely.

□

Corollary 4.2 *Assume that (3.1) and (4.1) are satisfied. Then the conditions (A), (B), (C) are satisfied.*

Proof: Follows from Theorem 3.1 and Proposition 4.1.

□

We now prove our main result.

Theorem 4.3 *Assume that (3.1) and (4.1) are satisfied. Then $\{\varphi_n^*\}$ converges uniformly on $D \cup T$. It follows that π_α has a continuous extension to $D \cup T$. Furthermore*

$$\min_{z \in D \cup T} |\pi_\alpha(z)| > 0 \tag{4.8}$$

(*where π_α denotes the extended function*).

Proof: From the recurrence relation (2.17) follows

$$\frac{\varphi_m^*(z)}{\varphi_{m-1}^*(z)} = -\tau_m \frac{\kappa_m(1-\overline{\alpha_{m-1}}z)}{\kappa_{m-1}(1-\overline{\alpha_m}z)}[\overline{\varepsilon_m} + \overline{\delta_m}\frac{z-\alpha_{m-1}}{1-\overline{\alpha_{m-1}}z}\frac{\varphi_{m-1}(z)}{\varphi_{m-1}^*(z)}], \qquad m = 1,2,\ldots. \tag{4.9}$$

Multiplication of (4.9) for $m = 1,2,\ldots n$ together with (2.18) gives

$$\varphi_n^*(z) = \varphi_0^*(z)\prod_{m=1}^{n}\frac{(1-\overline{\alpha_{m-1}}z)}{(1-\overline{\alpha_m}z)}\frac{\kappa_m}{\kappa_{m-1}}(-\tau_m)[\overline{\varepsilon_m} + \overline{\delta_m}\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}], \tag{4.10}$$

hence

$$\varphi_n^*(z) = \frac{\kappa_n}{1-\overline{\alpha_n}z}\prod_{m=1}^{n}(-\tau_m)[\overline{\varepsilon_m} + \overline{\delta}_m\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}]. \tag{4.11}$$

We note that $\frac{\kappa_n}{1-\overline{\alpha_n}z}$ converges to $\frac{\kappa}{1-\overline{\alpha}\,z}$ (with $\kappa = \lim_n \kappa_n$) uniformly on $D \cup T$. We may write the product in (4.11) as

$$\begin{aligned}&\prod_{m=1}^{n}(-\tau_m)[\overline{\varepsilon_m} + \overline{\delta_m}\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}] \\ &= \prod_{m=1}^{n}\{1 + [(-\tau_m\overline{\varepsilon_m} - 1) - \tau_m\overline{\delta_m}\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}]\}.\end{aligned} \tag{4.12}$$

The function $\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}$ is analytic on $D \cup T$ (since $\alpha_{m-1} \in D$ and the zeros of $\varphi_{m-1}^*(z)$ are in E), and

$$\left|\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}\right| = 1 \quad \text{for} \quad z \in T. \tag{4.13}$$

Consequently

$$\left|\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}\right| \leq 1 \quad \text{for} \quad z \in D \cup T, \tag{4.14}$$

hence

$$\begin{aligned}&|(-\tau_m\overline{\varepsilon_m} - 1) - \tau_m\overline{\delta_m}\frac{(z-\alpha_{m-1})\varphi_{m-1}(z)}{(1-\overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}| \\ &\leq |1 + \tau_m\overline{\varepsilon_m}| + |\delta_m| \quad \text{for} \quad z \in D \cup T.\end{aligned} \tag{4.15}$$

The assumption (4.1) together with (4.15) implies that the product in (4.11) and hence $\{\varphi_n^*\}$ converges uniformly on $D \cup T$ to a continuous function, which on D coincides with π_α. The limit function is different from zero since all the factors in the infinite product are different from zero and a convergent infinite product is zero only if one of the factors are zero.

□

Remark It follows from (4.1) that $|1 + \tau_m\overline{\varepsilon_m}| + |\delta_n| < 1$ for sufficiently large m, say $m \geq M$. Hence for $m \geq M$:

$$|1 + (-\tau_m\overline{\varepsilon_m} - 1) - \tau_m\overline{\delta_m}\frac{(z - \alpha_{m-1})\varphi_{m-1}(z)}{(1 - \overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}| \geq |1 - [|1 + \tau_m\overline{\varepsilon_m}| + |\delta_m|]| \tag{4.16}$$

which means

$$|(-\tau_m)[\overline{\varepsilon_m} + \overline{\delta_m}\frac{(z - \alpha_{m-1})\varphi_{m-1}(z)}{(1 - \overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}]| \geq |1 - [|1 + \tau_m\overline{\varepsilon_m}| + |\delta_m|]|. \tag{4.17}$$

Consequently

$$\left|\prod_{m=M}^{\infty}(-\tau_m)[\overline{\varepsilon_m} + \overline{\delta_m}\frac{(z - \alpha_{m-1})\varphi_{m-1}(z)}{(1 - \overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}]\right| \geq \prod_{m=M}^{\infty}|\{1 - [|1 + \tau_m\overline{\varepsilon_m}| + |\delta_m|]\}| \tag{4.18}$$

for $z \in D \cup T$. Thus (recall (4.11))

$$\lim_{n\to\infty}|\varphi_n^*(z)| \geq \frac{\kappa}{1 - |\alpha|}|\prod_{m=1}^{M-1}(-\tau_m)[\overline{\varepsilon_m} + \overline{\delta_m}\frac{(z - \alpha_{m-1})(\varphi_{m-1}(z)}{(1 - \overline{\alpha_{m-1}}z)\varphi_{m-1}^*(z)}]|\prod_{m=M}^{\infty}\{1 - [|1 + \tau_m\overline{\varepsilon_m}| + |\delta_m|]\} \tag{4.19}$$

If $|1 + \tau_m\overline{\varepsilon_m}| + |\delta_m| < 1$ for all m, then

$$\lim_{n\to\infty}|\varphi_n^*(z)| \geq \frac{\kappa}{1 - |\alpha|}\prod_{m=1}^{\infty}\{1 - [|1 + \tau_m\overline{\varepsilon_m}| + |\delta_m|]\}. \tag{4.20}$$

Corollary 4.4 *Assume that*

$$\alpha_n = \alpha \quad \text{for all} \quad n \tag{4.21}$$

and that

$$\sum_{n=1}^{\infty} |\delta_n| < \infty. \tag{4.22}$$

Then $\{\varphi_n^*\}$ *converges uniformly on* $D \cup T$, *and*

$$\min_{z \in D \cup T} |\pi_\alpha(z)| > 0 \tag{4.23}$$

(where π_α *denotes the continuous limit function on* $D \cup T$*).*

Proof: It follows from (2.14), (2.20) and (4.21) that in this situation $\varepsilon_m = -\tau_m$, hence $\tau_m \overline{\varepsilon_m} = -|\tau_m|^2 = -1$. Condition (4.1) then reduces to (4.20), and the result follows from Theorem 4.3.

□

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