Orthogonal rational functions and interpolatory product rules on the unit circle.

III: Convergence of general sequences*

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Abstract

Let $\mathcal{R}$ be the space of rational functions with poles among $\{\alpha_k, 1/\bar{\alpha}_k\}_{k=1}^\infty$ with $|\alpha_k| < 1$, $k \geq 1$. We consider a sequence $\{\mathcal{R}_n\}_{n=0}^\infty$ of nested subspaces with $\cup_{n=0}^\infty \mathcal{R}_n = \mathcal{R}$. We continue our investigation of the convergence as $n \rightarrow \infty$ of quadrature rules which are exact in $\mathcal{R}_n$. In Part II we have discussed the convergence for a particular nesting of the subspaces $\mathcal{R}_n$. In this part we prove similar convergence results for a more general sequence of subspaces. By similar arguments as in part II, we can also here derive related results about the convergence of multipoint rational approximants to the Riesz-Herglotz transform associated with a complex measure.

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1 Introduction

We use $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$ to denote the unit circle, its interior and its exterior in the complex plane $\mathbb{C}$. We set $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\alpha = \{\alpha_1, \alpha_2, \dots\}$, be an arbitrary sequence in $\mathbb{D}$. We define the Blaschke factors

$$\zeta_k(z) = z_k \frac{\alpha_k - z}{1 - \bar{\alpha}_k z}, \quad z_k = \begin{cases} -1 & \text{if } \alpha_k = 0 \\ \bar{\alpha}_k / |\alpha_k| & \text{if } \alpha_k \neq 0 \end{cases} \quad (1.1)$$

and the finite Blaschke products

$$B_0 = 1, \quad B_k(z) = \zeta_1(z) \zeta_2(z) \cdots \zeta_k(z), \quad k = 1, 2, \dots$$

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Setting

$$\pi_0 = 1, \quad \omega_0 = 1, \quad \pi_n = \prod_{k=1}^n (1 - \bar{\alpha}_k z), \quad \omega_n = \prod_{k=1}^n (z - \alpha_k), \quad n = 1, 2, \dots$$

it is clear that

$$B_n(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)}, \quad \eta_n = \prod_{k=1}^n (-z_k). \quad (1.2)$$

These functions generate the spaces $\mathcal{L}_n = \text{span}\{B_0, B_1, \dots, B_n\}$. It is clear that

$$R \in \mathcal{L}_n \Leftrightarrow R(z) = P(z)/\pi_n(z) \quad \text{with} \quad P \in \Pi_n,$$

where Π_n is the set of polynomials of degree at most n .

For a complex function f , we define its substar conjugate f_* by $f_*(z) = \overline{f(1/\bar{z})}$ and let it be understood of course that then the domain must be such as to allow the definition. We set $\mathcal{L}_{n*} = \{f : f_* \in \mathcal{L}_n\} = \text{span}\{1, B_{1*}, B_{2*}, \dots, B_{n*}\}$. Hence, it follows that

$$R \in \mathcal{L}_{n*} \Leftrightarrow R(z) = Q(z)/\omega_n(z) \quad \text{with} \quad Q \in \Pi_n.$$

Furthermore, we set $\mathcal{L} = \cup_n \mathcal{L}_n$, $\mathcal{L}_* = \cup_n \mathcal{L}_{n*}$, $\mathcal{R} = \mathcal{L} + \mathcal{L}_*$, and for nonnegative integers p and q , we set

$$\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q = \left\{ \frac{P}{\omega_p \pi_q} : P \in \Pi_{p+q} \right\}.$$

When all $\alpha_k = 0$, then $\mathcal{R}_{p,q} = \Lambda_{-p,q}$, the set of Laurent polynomials of the form $z^{-p}P(z)$ with $P \in \Pi_{p+q}$.

If for each $n = 0, 1, 2, \dots$ we define $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$ for some nonnegative integers $p(n)$ and $q(n)$, then there are infinitely many ways to define these $p(n)$ and $q(n)$ such that $\cup_n \mathcal{R}_n = \mathcal{R}$. We are especially interested in the case of nested subspaces, i.e. $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ and we assume that $\dim \mathcal{R}_n = n + 1$, thus $p(n) + q(n) = n$.

In [9], we have studied the following two special cases of nestings. Either

$$\mathcal{R}_{0,0} \subset \mathcal{R}_{0,1} \subset \mathcal{R}_{1,1} \subset \mathcal{R}_{1,2} \subset \mathcal{R}_{2,2} \subset \dots \quad (1.3)$$

i.e., $p(n) = q(n) = n/2$ when n is even and $(p(n), q(n)) = ((n-1)/2, (n+1)/2)$ when n is odd, thus $(p(n), q(n)) = (\lfloor n/2 \rfloor, \lceil n/2 \rceil)$. In this paper we use $\lfloor x \rfloor$ to denote the largest integer $\leq x$ and $\lceil x \rceil$ to denote the smallest integer $\geq x$. Or

$$\mathcal{R}_{0,0} \subset \mathcal{R}_{1,0} \subset \mathcal{R}_{1,1} \subset \mathcal{R}_{2,1} \subset \mathcal{R}_{2,2} \subset \dots \quad (1.4)$$

i.e., $p(n) = q(n) = n/2$ when n is even and $(p(n), q(n)) = ((n+1)/2, (n-1)/2)$ when n is odd, thus $(p(n), q(n)) = (\lceil n/2 \rceil, \lfloor n/2 \rfloor)$.

In this paper we study more general sequences $(p(n), q(n))$ than the ones described above. It is shown that the same kind of results as in [9] can be obtained in this more general situation.

So we assume that $\{p(n)\}$ and $\{q(n)\}$ are nondecreasing sequences of nonnegative integers, $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$ with $p(n) + q(n) = n$ so that $\mathcal{R}_n \subset \mathcal{R}_{n+1}$. This is assumed throughout the whole paper. In Section 5, where we discuss the rate of convergence of certain rational approximants and quadrature formulas, we assume that moreover

$$\lim_{n \rightarrow \infty} \frac{p(n)}{n} = r \in (0, 1). \quad (1.5)$$

Note that this implies that $\lim_{n \rightarrow \infty} p(n) = \lim_{n \rightarrow \infty} q(n) = \infty$ and that $\lim_{n \rightarrow \infty} [q(n)]/n = s = 1 - r \in (0, 1)$. The case considered in Part II is a special case corresponding to $r = s = 1/2$, and if moreover all $\alpha_k = 0$, we are in a polynomial situation as discussed by Thron in [17].

The nesting considered in Part II had a clear alternating pattern and this made us introduce the spaces $\tilde{\mathcal{L}}_n$ which were defined as $\mathcal{L}_n$, but with another sequence of points $\tilde{\alpha}$ which basically duplicated the points in α . Since this nice alternating pattern is lost in our more general case, the situation here will be somewhat more complicated.

Let us define

$$A_n = \{\alpha_1, \dots, \alpha_n\}$$

For $n = 1, 2, \dots$, set $r(n) = \max\{p(n), q(n)\}$ and $s(n) = \min\{p(n), q(n)\}$ and

$$\begin{aligned} \tilde{A}_n &= \{\alpha_1, \alpha_1, \dots, \alpha_{s(n)}, \alpha_{s(n)}, \alpha_{s(n)+1}, \alpha_{s(n)+2}, \dots, \alpha_{r(n)}\} \\ &= \{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n\}. \end{aligned}$$

Observe that there are $2s(n) + r(n) - s(n) = n$ points in $\tilde{A}_n$. Since $0 \leq p(n+1) - p(n) \leq 1$ and $0 \leq q(n+1) - q(n) \leq 1$ it follows that $s(n) \leq s(n+1)$ and $r(n) \leq r(n+1)$. This implies that $\tilde{A}_n \subset \tilde{A}_{n+1}$. The notation $\tilde{\alpha}_n$ means that $\tilde{A}_n = \tilde{A}_{n-1} \cup \{\tilde{\alpha}_n\}$. Thus $\tilde{\alpha} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n, \dots\}$ represents a new table of points in $\mathbb{D}$ which is a Newtonian table.

All quantities related to $\tilde{\alpha}$ will be denoted with a tilde. For example, we let $\tilde{\zeta}_n$ be the Blaschke factors for the table $\tilde{\alpha}$ and $\tilde{B}_n$ be the corresponding Blaschke products. As before, we define the spaces $\tilde{\mathcal{L}}_n = \text{span}\{1, \tilde{B}_1, \dots, \tilde{B}_n\}$, $\tilde{\mathcal{L}}_{n*} = \text{span}\{1, \tilde{B}_{1*}, \dots, \tilde{B}_{n*}\}$ and set $\tilde{\mathcal{L}} = \cup_{n=0}^{\infty} \tilde{\mathcal{L}}_n$, $\tilde{\mathcal{L}}_* = \cup_{n=0}^{\infty} \tilde{\mathcal{L}}_{n*}$, $\tilde{\mathcal{R}}_{m,n} = \tilde{\mathcal{L}}_{m*} + \tilde{\mathcal{L}}_n$ and $\tilde{\mathcal{R}} = \tilde{\mathcal{L}}_* + \tilde{\mathcal{L}}$.

For these spaces, we need a positive definite inner product, which as in part II, we define by

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} \omega(\theta) d\theta, \quad t = e^{i\theta} \quad (1.6)$$

where ω a positive weight function integrable on $[-\pi, \pi]$.

The outline of the paper is as follows.

In Section 2 we recall some preliminary results and in Section 3 we construct an orthogonal basis for $\mathcal{R}_n$ in terms of an orthogonal basis for $\tilde{\mathcal{L}}_n$. This generalizes some results of Part I.

In Section 4 we give several convergence results. First we describe quadrature formulas $I_n(f)$ which approximate integrals of the form

$$I_\rho(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \rho(\theta) d\theta$$

where ρ is a complex function, integrable on $[-\pi, \pi]$. Again, we could have used an arbitrary complex measure μ on $[-\pi, \pi]$. However, again for simplicity, we assume that $d\mu(\theta) = \rho(\theta) d\theta$

These quadrature formulas are product rules whose nodes are the zeros of para-orthogonal functions in $\tilde{\mathcal{L}}_n$ (with respect to ω and $\tilde{\alpha}$) and they are exact in the spaces $\mathcal{R}_{n-1}$. Then it is proved that under suitable conditions for ρ , ω and α , these $I_n(f)$ converge to $I_\rho(f)$ for any bounded function f for which the integral $I_\rho(f)$ exists.

Furthermore, it is well known that if the Riesz-Herglotz transform F_ρ of the measure $d\mu(\theta) = \rho(\theta) d\theta$ is defined as in [1]

$$F_\rho(z) = \int_{-\pi}^{\pi} D(t, z) \rho(\theta) d\theta, \quad D(t, z) = \frac{t + z}{t - z}, \quad t = e^{i\theta}, \quad (1.7)$$

then $F_n(z) = I_n(D(\cdot, z))$ provides a multipoint Padé-type approximant (MPTA) to $F_\rho(z)$. It will follow from the previous convergence result that F_n converges to F_ρ uniformly on compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$.

Finally, it is shown in Section 5 that the rate of convergence is geometric both for the quadrature formulas and for the MPTA under certain restrictive conditions on ω and α .

2 Preliminaries

Let $\{\phi_n\}_{n=0}^\infty$ be an orthogonal basis for $\mathcal{L}$, obtained by orthogonalizing the basis $\{1, B_1, B_2, \dots\}$ with respect to ω . Thus $\phi_n \in \mathcal{L}_n - \mathcal{L}_{n-1}$ and $\phi_n \perp \mathcal{L}_{n-1}$. It is easily checked that $\phi_n^* \perp \zeta_n \mathcal{L}_{n-1}$. If we define $\phi_n^* = B_n \phi_{n*}$, then it is known that

$$\chi_n = \phi_n + \delta_n \phi_n^*, \quad \delta_n \in \mathbb{T}$$

are para-orthogonal functions, i.e., they are orthogonal to $\mathcal{L}_{n-1} \cap \zeta_n \mathcal{L}_{n-1}$. We refer to it as para-orthogonal functions with respect to the weight ω and the table α . Moreover, these functions χ_n have n distinct zeros on $\mathbb{T}$. If we take these zeros as nodes in the interpolatory quadrature formula, which is exact in $\mathcal{R}_{n-1, n-1}$, then the quadrature formula is called a rational Szegő quadrature formula. Its weights are known to be positive. It always exists and is uniquely defined when the parameters δ_n in the para-orthogonal functions are chosen so that the nodes are fixed. Furthermore, if $\sum_{k=1}^\infty (1 - |\alpha_k|) = \infty$, then the rational Szegő formulas will converge to the exact integral for any bounded Riemann integrable function f . This was proved in [5].

Obviously, the whole theory concerning orthogonal rational functions on $\mathbb{T}$ for the table α can be repeated for the table $\tilde{\alpha}$. Note that $\sum_{k=1}^\infty (1 - |\alpha_k|) = \infty$ implies that also $\sum_{k=1}^\infty (1 - |\tilde{\alpha}_k|) = \infty$. This immediately gives us the following result about rational Szegő formulas.

Theorem 2.1 *Let ψ be a finite positive measure on $[-\pi, \pi]$. We consider spaces with a tilde, i.e., spaces based on the points $\tilde{\alpha}$. Then*

(1) *For each $n = 1, 2, \dots$, there exist n distinct points $\{x_{j,n}\}_{j=1}^n$ on $\mathbb{T}$ and n positive numbers $\{w_{j,n}\}_{j=1}^n$ such that*

$$\tilde{Q}_n(f) := \sum_{j=1}^n w_{j,n} f(x_{j,n}) = I_\psi(f) := \int_{-\pi}^{\pi} f(e^{i\theta}) d\psi(\theta), \quad \forall f \in \tilde{\mathcal{R}}_{n-1, n-1}.$$

(2) *If $\sum_{k=1}^\infty (1 - |\alpha_k|) = \infty$, then*

$$\lim_{n \rightarrow \infty} \tilde{Q}_n(f) = I_\psi(f),$$

for all bounded Riemann integrable functions f on $\mathbb{T}$.

To prove the rates of convergence in Section 5, we need to know something about the distribution of the α 's. Let ν_n^α be the normalized counting measure which assigns a point mass to α_j , taking the multiplicity of α_j into account. Let $\hat{\alpha} = \{1/\bar{\alpha}_k : \alpha_k \in \alpha\}$ and let $\nu_n^{\hat{\alpha}}$ be defined correspondingly. The following lemma gives the n th root asymptotics of the Blaschke products when $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$, i.e., when ν_n^α converges to some measure ν^α in weak star topology. It was proved in [7, Lemma 4.6].

Lemma 2.2 Let B_n denote the Blaschke products with zeros $\{\alpha_1, \dots, \alpha_n\}$. Suppose that we have for the zero distribution $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. Then

$$\lim_{n \rightarrow \infty} |B_n(z)|^{1/n} = \exp\{\lambda(z)\} \quad \text{and} \quad \lim_{n \rightarrow \infty} |B_n(z)|^{-1/n} = \exp\{\lambda(\hat{z})\}$$

with $\hat{z} = 1/\bar{z}$ and

$$\lambda(z) = \int \log |\zeta_z(x)| d\nu^\alpha(x), \quad \zeta_z(x) = \frac{x - z}{1 - \bar{z}x} \quad (2.1)$$

and this holds uniformly for $z \in \mathbb{C} - \{\{0\} \cup \text{supp}(\nu^\alpha) \cup \text{supp}(\nu^{\hat{\alpha}})\}$.

Note that $\lambda(z) = -\lambda(\hat{z})$.

We also require the n th root asymptotics for the para-orthogonal rational functions

$$\chi_n(z) = \phi_n(z) + \delta_n \phi_n^*(z), \quad \delta_n \in \mathbb{T}.$$

We have (see [7, Theorems 4.8-9])

Theorem 2.3 Suppose $\int_{-\pi}^{\pi} \log \omega(\theta) d\theta > -\infty$, $\sum(1 - |\alpha_k|) = \infty$ and $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. Let χ_n be the para-orthogonal functions with respect to ω and α . Then the following holds locally uniformly in the indicated regions

- (1) $\lim_{n \rightarrow \infty} |\chi_n(z)|^{1/n} = 1, \quad z \in \mathbb{D}$
- (2) $\lim_{n \rightarrow \infty} |\chi_n(z)|^{1/n} = \exp\{\lambda(z)\}, \quad z \in \mathbb{E} - \text{supp}(\nu^{\hat{\alpha}})$
- (3) $\limsup_{n \rightarrow \infty} |\chi_n(z)|^{1/n} \leq \exp\{\lambda(z)\}, \quad z \in \mathbb{E}$

where λ is as in (2.1).

Furthermore, one has

$$\lim_{n \rightarrow \infty} \|\chi_n\|_\infty^{1/n} = 1 \quad (2.2)$$

where

$$\|\chi_n\|_\infty = \max_{x \in \mathbb{T}} \{|\chi_n(x)|\} = \max_{x \in \mathbb{D} \cup \mathbb{T}} \{|\chi_n(x)|\}.$$

We end these preliminaries with the following lemma which will be required later.

Lemma 2.4 Let f and g be functions in $\mathcal{R}_n = \mathcal{R}_{p(n), q(n)}$. Then

$$fg_* \in \tilde{R}_{n,n} = \tilde{\mathcal{L}}_{n*} + \tilde{\mathcal{L}}_n.$$

Proof. Take $f, g \in \mathcal{R}_n$, then

$$f(z) = \frac{P(z)}{\omega_{p(n)}(z)\pi_{q(n)}(z)}, \quad g(z) = \frac{Q(z)}{\omega_{p(n)}\pi_{q(n)}(z)}, \quad P, Q \in \Pi_n.$$

Hence

$$g_*(z) = \frac{Q^*(z)}{\pi_{p(n)}(z)\omega_{q(n)}(z)}, \quad Q^*(z) = z^n Q_*(z) \in \Pi_n,$$

and therefore

$$f(z)g_*(z) = \frac{T(z)}{\omega_{p(n)}(z)\omega_{q(n)}(z)\pi_{p(n)}(z)\pi_{q(n)}(z)}, \quad T = PQ^* \in \Pi_{2n}.$$

Thus setting $s = \min\{p(n), q(n)\}$ and $r = \max\{p(n), q(n)\}$, gives

$$(fg_*)(z) = \frac{T(z)}{\omega_s^2(z)\pi_s^2(z)\prod_{j=s+1}^r(z-\alpha_j)(1-\bar{\alpha}_jz)} = \frac{T(z)}{\tilde{\omega}_n(z)\tilde{\pi}_n(z)}, \quad T \in \Pi_{2n}.$$

and this proves the lemma. $\square$

Observe that in particular $\mathcal{R}_n \subset \tilde{\mathcal{R}}_{n,n}$. Also note that when all $\alpha_k = 0$, then $\alpha = \tilde{\alpha} = \{0, 0, \dots\}$ and $\mathcal{R}_n = \Lambda_{-p(n), q(n)}$ and $\tilde{\mathcal{R}}_{n,n} = \Lambda_{-n,n}$.

3 Orthogonal basis

In this section we give an explicit orthogonal basis for the spaces

$$\mathcal{R}_n := \mathcal{R}_{p(n), q(n)} = \mathcal{L}_{p(n)*} + \mathcal{L}_{q(n)}, \quad n = 0, 1, 2, \dots$$

Orthogonality is with respect to the inner product (1.6). We look for functions σ_n , such that $\mathcal{R}_{n-1} \perp \sigma_n \in \mathcal{R}_n - \mathcal{R}_{n-1}$, $n = 1, 2, \dots$. We need this in the next section. To obtain such an orthogonal basis, it is sufficient to apply the Gram-Schmidt process to the basis $B_{p(n)*} = 1/B_{p(n)}, \dots, B_0 = 1, \dots, B_{q(n)}$ in the order dictated by the sequences $\{p(k)\}_{k=0}^n$ and $\{q(k)\}_{k=0}^n$. However we derive like in Part II, a basis for $\mathcal{R}_n$ from the orthogonal basis for the spaces $\tilde{\mathcal{L}}_n$.

Theorem 3.1 *Let $\{p(n)\}$ and $\{q(n)\}$ be nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$, so that $\dim \mathcal{R}_n = n + 1$. Let $\tilde{\phi}_n \in \tilde{\mathcal{L}}_n - \tilde{\mathcal{L}}_{n-1}$ be the n th orthogonal function with respect to the inner product (1.6). Define $\sigma_0 = 1$ and for $n = 1, 2, \dots$*

$$\begin{aligned} \sigma_n(z) &= B_{p(n)*}(z)\tilde{\phi}_n(z) \quad \text{if } p(n) = p(n-1) \\ \sigma_n(z) &= B_{p(n)*}(z)\tilde{\phi}_n^*(z) \quad \text{if } p(n) > p(n-1). \end{aligned}$$

Then $\sigma_n \in \mathcal{R}_n - \mathcal{R}_{n-1}$ and $\sigma_n \perp \mathcal{R}_{n-1}$ for $n = 1, 2, \dots$.

Proof. We first note that $\tilde{\omega}_n(z) = \prod_{j=1}^n(z - \tilde{\alpha}_j) = \omega_{s(n)}(z)\omega_{r(n)}(z) = \omega_{p(n)}(z)\omega_{q(n)}(z)$ and $\tilde{\pi}_n(z) = \prod_{j=1}^n(1 - \tilde{\alpha}_jz) = \pi_{p(n)}(z)\pi_{q(n)}(z)$. Also $B_p = \eta_p\omega_p/\pi_p$ where η_p is as in (1.2). Since both $\tilde{\phi}_n$ and $\tilde{\phi}_n^*$ belong to $\tilde{\mathcal{L}}_n$ one has in both cases that σ_n is of the form

$$\sigma_n(z) = B_{p(n)*}(z)P_n(z)/\tilde{\pi}_n(z), \quad P_n \in \Pi_n$$

and thus

$$\sigma_n(z) = \bar{\eta}_{p(n)} \frac{\pi_{p(n)}(z)P_n(z)}{\omega_{p(n)}(z)\pi_{p(n)}(z)\pi_{q(n)}(z)} = \bar{\eta}_{p(n)} \frac{P_n(z)}{\omega_{p(n)}(z)\pi_{q(n)}(z)} \in \mathcal{R}_{p(n), q(n)}.$$

Assume first $p(n) = p(n-1)$ and hence $q(n) = q(n-1) + 1$, then we have to prove that $\sigma_n = B_{p(n)*}\tilde{\phi}_n$ is orthogonal to all $f \in \mathcal{R}_{n-1} = \mathcal{L}_{p(n-1)*} + \mathcal{L}_{q(n-1)}$.

Take first $f \in \mathcal{L}_{q(n-1)}$, then $f = P/\pi_{q(n-1)}$ with $P \in \Pi_{q(n-1)}$. Therefore

$$\langle \sigma_n, f \rangle = \langle B_{p(n)*}\tilde{\phi}_n, f \rangle = \langle \tilde{\phi}_n, fB_{p(n)} \rangle. \quad (3.1)$$

Note that

$$fB_{p(n)} = \eta_{p(n)} \frac{P\omega_{p(n)}}{\pi_{q(n-1)}\pi_{p(n)}} = \eta_{p(n)} \frac{P\omega_{p(n-1)}}{\pi_{q(n-1)}\pi_{p(n-1)}} = \frac{T}{\tilde{\pi}_{n-1}} \in \tilde{\mathcal{L}}_{n-1}$$

where $T \in \Pi_{n-1}$. Because $\tilde{\phi}_n \perp \tilde{\mathcal{L}}_{n-1}$ we find $\langle \sigma_n, f \rangle = 0$.

Next take $f \in \mathcal{L}_{p(n-1)*}$, then $f = P/\omega_{p(n-1)}$ with $P \in \Pi_{p(n-1)}$. Now $fB_{p(n)} \in \tilde{\mathcal{L}}_{n-1}$ because for the case $p(n) = p(n-1)$

$$fB_{p(n)} = \eta_{p(n)} \frac{P\omega_{p(n)}}{\omega_{p(n-1)}\pi_{p(n)}} = \eta_{p(n)} \frac{P}{\pi_{p(n-1)}} \in \tilde{\mathcal{L}}_{n-1}.$$

Again, we find from (3.1) that $\langle \sigma_n, f \rangle = 0$.

On the other hand, if $p(n) > p(n-1)$, then $q(n) = q(n-1)$. Since $\tilde{\phi}_n^* \perp \tilde{\zeta}_n \tilde{\mathcal{L}}_{n-1}$, we find that for $\sigma_n = B_{p(n)*} \tilde{\phi}_n^*$ and $f \in \mathcal{L}_{q(n)}$ i.e., $f = P/\pi_{q(n)}$ with $P \in \Pi_{q(n)}$, it holds that $\langle \sigma_n, f \rangle = \langle \tilde{\phi}_n^*, B_{p(n)} f \rangle$. Because

$$(fB_{p(n)})(z) = \eta_{p(n)} \frac{P(z)\omega_{p(n)}(z)}{\pi_{p(n)}(z)\pi_{q(n)}(z)} = \eta_{p(n)} \frac{(z - \alpha_{p(n)})\omega_{p(n-1)}(z)P(z)}{(1 - \bar{\alpha}_{p(n)}z)\pi_{p(n-1)}(z)\pi_{q(n-1)}(z)} \in \tilde{\zeta}_n \tilde{\mathcal{L}}_{n-1}$$

Hence $\langle \sigma_n, f \rangle = 0$. Finally, take $f \in \mathcal{L}_{p(n-1)*}$, i.e., $f = P/\omega_{p(n-1)}$ with $P \in \Pi_{p(n-1)}$ and $p(n) = p(n-1) + 1$. It remains to check that $B_{p(n)}f \in \tilde{\zeta}_n \tilde{\mathcal{L}}_{n-1}$. This is true since

$$(fB_{p(n)})(z) = \eta_{p(n)} \frac{P(z)\omega_{p(n)}(z)}{\pi_{p(n)}(z)\omega_{p(n-1)}(z)} = \eta_{p(n)} \frac{(z - \alpha_{p(n)})\omega_{p(n-1)}(z)P(z)}{(1 - \bar{\alpha}_{p(n)}z)\pi_{p(n-1)}(z)\omega_{p(n-1)}(z)} \in \tilde{\zeta}_n \tilde{\mathcal{L}}_{n-1}.$$

This concludes the proof. $\square$

Example 3.1 With the meaning of $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ as given before, let us take $p(n) = \lfloor n/2 \rfloor$, then $q(n) = n - p(n) = \lceil n/2 \rceil$. We have $\mathcal{R}_0 = \mathcal{R}_{0,0}$, $\mathcal{R}_1 = \mathcal{R}_{0,1}$, $\mathcal{R}_2 = \mathcal{R}_{1,1}$, $\mathcal{R}_3 = \mathcal{R}_{1,2}, \dots$. Thus we have the nesting (1.3). Assume n is even, say $n = 2m$, then $p(n) = m$ and $p(n-1) = m-1$. So $p(n) > p(n-1)$ and by Theorem 3.1 one has

$$\sigma_n(z) = \sigma_{2m}(z) = B_{p(n)*}(z)\tilde{\phi}_n^*(z) = B_{m*}(z)\tilde{\phi}_{2m}^*(z).$$

Take now n odd, i.e., $n = 2m + 1$, then $p(n) = m = p(n-1)$ and therefore

$$\sigma_{2m+1}(z) = B_{m*}(z)\tilde{\phi}_{2m+1}^*(z).$$

On the other hand, taking $p(n) = \lceil n/2 \rceil$ and hence $q(n) = n - p(n) = \lfloor n/2 \rfloor$, then we get the nesting (1.4). Since $p(n) = p(n-1)$ if n is even and $p(n) > p(n-1)$ when n is odd, we get in this case for the orthogonal basis

$$\begin{cases} \sigma_{2m} = B_{m*}(z)\tilde{\phi}_{2m}^* & n = 2m \\ \sigma_{2m+1} = B_{m*}(z)\tilde{\phi}_{2m+1}^* & n = 2m + 1. \end{cases}$$

These were the two choices that were considered in the first two parts of this paper [8, 9]. The same selection was also made in the polynomial case (i.e., all $\alpha_k = 0$) by Thron [17]. The quadrature formulas considered in those papers were exact in $\mathcal{R}_{2n} = \mathcal{R}_{p(2n), q(2n)}$, where in the two cases of this example we always have $p(2n) = q(2n) = n$.

In Gaussian quadrature formulas on an interval of the real line, one takes for the nodes the zeros of orthogonal polynomials, since these are known to be simple and located in the interval. When we are dealing with the unit circle, the orthogonal polynomials have all their zeros inside $\mathbb{D}$ and therefore the orthogonal polynomials on an interval are replaced by the para-orthogonal polynomials on the unit circle (or in our case para-orthogonal rational functions). These have indeed only simple zeros which are all in $\mathbb{T}$. In the sequel, we shall use as quadrature nodes the zeros of the para-orthogonal rational functions $\tilde{\chi}_n = \tilde{\phi}_n + \delta_n \tilde{\phi}_n^*$ where $\delta_n \in \mathbb{T}$ and the $\tilde{\phi}_n$ are the orthogonal rational functions in $\tilde{\mathcal{L}}_n$. To give the relation with the orthogonal basis σ_n , we note that

$$\tilde{\chi}_n = \tilde{\phi}_n + \delta_n \tilde{\phi}_n^* = \begin{cases} B_{p(n)}\sigma_n + \delta_n B_{q(n)}\sigma_{n*} & \text{if } p(n) = p(n-1) \\ B_{q(n)}\sigma_{n*} + \delta_n B_{p(n)}\sigma_n & \text{if } p(n) > p(n-1). \end{cases}$$

More precisely, we can formulate the following.

Theorem 3.2 *Let $\{\sigma_n\}_0^\infty$ be an orthogonal basis for $\mathcal{R}$, such that $\sigma_n \in \mathcal{R}_n - \mathcal{R}_{n-1}$. Set*

$$\sigma_n(z) = \frac{P_n(z)}{\omega_{p(n)}(z)\pi_{q(n)}(z)}, \quad P_n \in \Pi_n.$$

and take $\delta_n \in \mathbb{T}$. Then the zeros $\{x_{j,n}\}$ of the para-orthogonal rational functions $\tilde{\chi}_n = \tilde{\phi}_n + \delta_n \tilde{\phi}_n^$ are the zeros of the polynomials*

$$Q_n(z) = \eta_{p(n)}P_n(z) + \delta_n \eta_{q(n)}P_n^*(z), \quad \text{if } p(n) = p(n-1) \quad (3.2)$$

$$Q_n(z) = \eta_{p(n)}P_n(z) + \bar{\delta}_n \eta_{q(n)}P_n^*(z), \quad \text{if } p(n) > p(n-1) \quad (3.3)$$

where η_k is as in (1.2). This polynomial Q_n is para-orthogonal with respect to the varying measure

$$d\psi_n(\theta) = \frac{d\psi(\theta)}{|\omega_{p(n-1)}(x)\omega_{q(n-1)}(x)|^2}, \quad x = e^{i\theta}.$$

Proof. Set

$$\sigma_n(z) = \frac{P_n(z)}{\omega_{p(n)}(z)\pi_{q(n)}(z)}, \quad p(n) + q(n) = n, \quad n = 0, 1, \dots$$

Assume $p(n) = p(n-1)$. Then $\sigma_n = B_{p(n)*}\tilde{\phi}_n$. Thus

$$\tilde{\phi}_n = B_{p(n)}\sigma_n = \eta_{p(n)} \frac{\omega_{p(n)}P_n}{\pi_{p(n)}\omega_{p(n)}\pi_{q(n)}} = \eta_{p(n)} \frac{P_n}{\tilde{\pi}_n}.$$

Furthermore $\tilde{\phi}_n^* = \eta_{q(n)}P_n^*/\tilde{\pi}_n$ and therefore

$$\tilde{\chi}_n = \tilde{\phi}_n + \delta_n \tilde{\phi}_n^* = \frac{\eta_{p(n)}P_n + \delta_n \eta_{q(n)}P_n^*}{\tilde{\pi}_n}$$

Since the nodes $\{x_{j,n}\}_{j=0}^n$ are the zeros of $\tilde{\chi}_n$, the first part follows in the case $p(n) = p(n-1)$.

If $p(n) > p(n-1)$, then $\sigma_n = B_{p(n)*}\tilde{\phi}_n^*$, thus $\tilde{\phi}_n^* = B_{p(n)}\sigma_n$ and consequently $\tilde{\phi}_n^* = \eta_{p(n)}P_n/\tilde{\pi}_n$. Thus $\tilde{\phi}_n = \eta_{q(n)}P_n^*/\tilde{\pi}_n$ and so

$$\tilde{\chi}_n = \tilde{\phi}_n + \delta_n \tilde{\phi}_n^* = \delta_n \frac{\eta_{p(n)}P_n + \bar{\delta}_n \eta_{q(n)}P_n^*}{\tilde{\pi}_n}.$$

And this concludes the first part.

It remains to be shown that the polynomials are para-orthogonal with respect to ψ_n . Therefore, we prove that their zeros $\{x_{j,n}\}$ generate a quadrature formula which is exact in $\Lambda_{-(n-1),n-1}$ for integrals of the form $\int f d\psi_n$. By Theorem 6 in [3] (see also [11, Theorem 7.1]), it then follows that the polynomials Q_n are para-orthogonal with respect to this measure ψ_n .

To prove this, we take $L \in \Lambda_{-(n-1),n-1}$ and set $L(z) = z^{-n+1}P(z)$ with $P \in \Pi_{2n-2}$. Then (set $x = e^{i\theta}$)

$$\begin{aligned} \int_{-\pi}^{\pi} L(x) d\psi_n(x) &= \int_{-\pi}^{\pi} \frac{P(x) d\psi(\theta)}{x^{n-1} |\omega_{p(n-1)}(x) \omega_{q(n-1)}(x)|^2} \\ &= \int_{-\pi}^{\pi} \frac{P(x)}{\tilde{\pi}_{n-1}(x) \tilde{\omega}_{n-1}(x)} d\psi(\theta). \end{aligned}$$

Call the integrand $R(x)$, then obviously $R \in \tilde{\mathcal{R}}_{n-1,n-1}$. If the rational Szegő quadrature formula with respect to the measure ψ and the table $\tilde{\alpha}$ is denoted as $\tilde{Q}_n(f) = \sum w_{j,n} f(x_{j,n})$ (note that its nodes are precisely the $\{x_{j,n}\}_{j=1}^n$), then since it is exact in $\tilde{\mathcal{R}}_{n-1,n-1}$, we get

$$\int_{-\pi}^{\pi} L(x) d\psi_n(x) = \sum_{j=1}^n w_{j,n} R(x_{j,n}).$$

Because $R(x) = x^{n-1} L(x) / [\tilde{\omega}_{n-1}(x) \tilde{\pi}_{n-1}(x)]$, one has that for any $L \in \Lambda_{-(n-1),n-1}$

$$\int_{-\pi}^{\pi} L(x) d\psi_n(\theta) = \sum_{j=1}^n A_{j,n} L(x_{j,n}),$$

with

$$A_{j,n} = \frac{x_{j,n}^{n-1} w_{j,n}}{\tilde{\omega}_{n-1}(x_{j,n}) \tilde{\pi}_{n-1}(x_{j,n})} = \frac{w_{j,n}}{|\omega_{p(n-1)}(x_{j,n}) \omega_{q(n-1)}(x_{j,n})|^2} > 0.$$

Thus the quadrature formula $\sum_{j=1}^n A_{j,n} L(x_{j,n})$ is exact in $\Lambda_{-(n-1),n-1}$ with positive weights, thus it must be a Szegő formula for the measure ψ_n and hence its nodes are the zeros of para-orthogonal polynomials with respect to this measure. This proves the theorem. $\square$

4 Convergence

In this section we prove convergence results for quadrature formulas for $I_\rho(f)$ and for multipoint Padé-type approximants to the Riesz-Herglotz transform F_ρ . We start with the quadrature formulas.

We consider formulas of the form

$$I_n(f) := \sum_{j=1}^n A_{j,n} f(x_{j,n}) \approx I_\rho(f) := \int_{-\pi}^{\pi} f(e^{i\theta}) \rho(\theta) d\theta \quad (4.1)$$

where ρ is a complex valued function.

Let $\mathbb{X} = \{x_{j,n} : j = 1, 2, \dots, n; n = 1, 2, \dots\}$ be a triangular array of distinct points on $\mathbb{T}$. Then it is known [5] that there exist uniquely defined (in general not positive) weights $A_{j,n}$ (depending on ρ) such that

$$I_n(f) = I_\rho(f), \quad \forall f \in \mathcal{R}_{n-1} := \mathcal{R}_{p(n-1),q(n-1)}. \quad (4.2)$$

These quadrature formulas are interpolatory quadrature formulas since they can be obtained by integrating an interpolant from $\mathcal{R}_{n-1}$ for f . Thus, $I_n(f) = I_\rho(L_n^f)$ where L_n^f is the unique function in $\mathcal{R}_{n-1}$ satisfying the interpolation conditions

$$L_n^f(x_{j,n}) = f(x_{j,n}), \quad j = 1, 2, \dots, n. \quad (4.3)$$

Recall that $\dim \mathcal{R}_{n-1} = n$ which implies that the weights are indeed uniquely defined.

Since the measure $d\mu(\theta) = \rho(\theta)d\theta$ is complex, we can not use it to construct Szegő quadrature formulas. Indeed, then we need to choose the quadrature nodes as the zeros of para-orthogonal functions, but since the measure is complex, we do not have the right framework to construct the nodes. In order to give an appropriate choice for the array $\mathbb{X}$, we have to consider an auxiliary positive measure $d\psi$ on $[-\pi, \pi]$. We construct for this measure (and for the table $\tilde{\alpha}$) the orthogonal functions $\{\tilde{\phi}_n\}$ for the spaces $\tilde{\mathcal{L}}$ obtained by orthogonalizing the basis $\{1, \tilde{B}_1, \tilde{B}_2, \dots\}$. We then choose $\delta_n \in \mathbb{T}$ arbitrary unimodular complex numbers and define the para-orthogonal rational functions (with respect to the measure ψ and table $\tilde{\alpha}$)

$$\tilde{\chi}_n := \tilde{\phi}_n + \delta_n \tilde{\phi}_n^*, \quad \tilde{\phi}_n \in \tilde{\mathcal{L}}_n - \tilde{\mathcal{L}}_{n-1}, \quad \tilde{\phi}_n \perp \tilde{\mathcal{L}}_{n-1}. \quad (4.4)$$

Since the function $\tilde{\chi}_n$ has exactly n distinct zeros on $\mathbb{T}$, we can choose them as the quadrature nodes $\{x_{j,n}\}_{j=1}^n$. We shall prove the convergence of the quadrature formulas $I_n(f)$ when the table $\mathbb{X}$ is chosen to contain these nodes. Note that the interpolatory quadratures $I_n(f)$ are not rational Szegő formulas because there are two measures involved: the complex measure μ for the integral and an auxiliary positive measure ψ to define the nodes. Since the measure ψ is only used to define these nodes, it is not important to consider the most general case and it is thus not really a restriction to only consider absolutely continuous measures $d\psi(\theta) = \omega(\theta)d\theta$ with $\omega(\theta) > 0$ a.e. on $[-\pi, \pi]$. We only need the existence of such an ω . Occasionally, we also need the rational Szegő formula for the weight ω and the table $\tilde{\alpha}$. We shall denote it as before by $\tilde{Q}_n(f)$. Note that it has positive weights and it holds that

$$\tilde{Q}_n(f) := \sum_{j=1}^n w_{j,n} f(x_{j,n}) = I_\omega(f) := \int_{-\pi}^{\pi} f(e^{i\theta}) \omega(\theta) d\theta, \quad \forall f \in \tilde{R}_{n-1, n-1}. \quad (4.5)$$

To prove the convergence of $I_n(f)$ with this particular choice of the nodes, we have to prove that the interpolant L_n^f converges to f in $L_\omega^2(\mathbb{T})$.

Theorem 4.1 *Let $\{x_{j,n}\}_{j=1}^n$ be the zeros of the para-orthogonal rational function $\tilde{\chi}_n$ (with respect to ω and $\tilde{\alpha}$) and assume that $\sum_{j=1}^{\infty} (1 - |\alpha_j|) = \infty$. Then the sequence of interpolants $\{L_n^f\}$ at the points $\{x_{j,n}\}$ converges to f in $L_{\omega,2}$ -norm, i.e.,*

$$\lim_{n \rightarrow \infty} \|L_n^f - f\|_{\omega,2}^2 := \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} |L_n^f(e^{i\theta}) - f(e^{i\theta})|^2 \omega(\theta) d\theta = 0$$

for any function f bounded on $\mathbb{T}$ for which the Riemann integral $I_\omega(f)$ exists.

Proof. Since f is a bounded Riemann integrable function and because $\mathcal{R}$ is dense in $C(\mathbb{T})$, the space of continuous functions on $\mathbb{T}$, [5, Lemma 3], and because $C(\mathbb{T})$ is dense in any L_ω^2 , there is a function $S \in \mathcal{R}$ such that $\|f - S\|_{\omega,2} < \epsilon$ for any given $\epsilon > 0$. Obviously, the function $g = f - S$ is bounded and Riemann integrable.

Let L_n^g be the unique interpolant from $\mathcal{R}_{n-1}$ that interpolates g in the points $\{x_{j,n}\}_{j=1}^n$. We compute the norm (we set $x = e^{i\theta}$)

$$\|L_n^g - g\|_{\omega,2}^2 = \int_{-\pi}^{\pi} |L_n^g(x) - g(x)|^2 \omega(\theta) d\theta = \|g\|_{\omega,2}^2 + \|L_n^g\|_{\omega,2}^2 - 2I_{\omega}(\operatorname{Re}(\bar{g}L_n^g)).$$

Since $L_n^g \in \mathcal{R}_{n-1}$, we find that $|L_n^g|^2 = L_n^g L_{n*}^g \in \tilde{\mathcal{R}}_{n-1,n-1}$. Hence

$$I_{\omega}(|L_n^g|^2) = \tilde{Q}_n(|L_n^g|^2) = \sum_{j=1}^n w_{j,n} |L_n^g(x_{j,n})|^2 = \tilde{Q}_n(|g|^2).$$

Furthermore, by the Cauchy-Schwarz inequality

$$\begin{aligned} |I_{\omega}\{\operatorname{Re}(\bar{g}L_n^g)\}|^2 &= \left| \int_{-\pi}^{\pi} \operatorname{Re}(\bar{g}(x)L_n^g(x)) \omega(\theta) d\theta \right|^2, \quad x = e^{i\theta} \\ &\leq \left[\int_{-\pi}^{\pi} |\operatorname{Re}(\bar{g}(x)L_n^g(x))| \omega(\theta) d\theta \right]^2 \\ &\leq \left[\int_{-\pi}^{\pi} |g(x)L_n^g(x)| \omega(\theta) d\theta \right]^2 \\ &\leq \|g\|_{\omega,2}^2 \|L_n^g\|_{\omega,2}^2 = I_{\omega}(|g|^2) \tilde{Q}_n(|g|^2). \end{aligned}$$

Since by Lemma 2.4 and Theorem 2.1 $\lim_{n \rightarrow \infty} I_n(|g|^2) = I_{\omega}(|g|^2)$, it follows that

$$\limsup_{n \rightarrow \infty} \|g - L_n^g\|_{\omega,2}^2 \leq 4\|g\|_{\omega,2}^2.$$

For sufficiently large n , we have $L_n^g = L_n^f - S$. Thus

$$\limsup_{n \rightarrow \infty} \|f - L_n^f\|_{\omega,2}^2 \leq \limsup_{n \rightarrow \infty} \|(f - S) - L_n^{(f-S)}\|_{\omega,2}^2 \leq 4\epsilon.$$

Since ϵ is arbitrary small, the theorem follows. $\square$

Remark 4.1 In the previous part of this paper [9, Lemma 4.4], we proved this theorem for the particular case $p(2n+1) = n$, $n = 1, 2, \dots$ and for the particular nestings (1.3) and (1.4).

Remark 4.2 This theorem generalizes the classical Erdős-Turán Theorem [10]. For other extensions see also [2, 12].

Remark 4.3 When all the α_k are equal to zero, we are dealing with Laurent polynomials on $\mathbb{T}$, i.e., with trigonometric polynomials. We can illustrate this with a classical result from trigonometric interpolation. Suppose $\alpha = \tilde{\alpha} = \{0, 0, \dots\}$ so that for all m and $n \geq 0$ $\mathcal{R}_{m,n} = \tilde{\mathcal{R}}_{m,n} = \Lambda_{-m,n}$. Take $p(2n+1) = q(2n+1) = n$ and define $N = 2n+1$. Then for a given weight function $\omega(\theta)$, let ϕ_N be the N th Szegő polynomial and define the para-orthogonal polynomial $\chi_N(z) = \phi_N(z) + \delta_N \phi_N^*(z)$. Let $\{x_{j,N} = e^{i\theta_j}\}_{j=1}^N$ be the zeros of χ_N . Then for the trigonometric polynomial $T_n(\theta) = \sum_{j=0}^n [a_j \cos(j\theta) + ib_j \sin(j\theta)] = L_n^f(e^{i\theta})$ satisfying $T_n(\theta_j) = f(\theta_j)$, for $j = 1, 2, \dots, N$; $N = 2n+1$, it holds that

$$\lim_{n \rightarrow \infty} \|T_n - f\|_{\omega,2} = 0. \quad (4.6)$$

When we take $\omega(\theta) = 1$, then $\phi_N(z) = z^N$ and $\phi_N^*(z) = 1$, so that by choosing $\delta_N = -1$, one has $\chi_N(z) = z^N - 1$, i.e., the $\{x_{j,N}\}$ are the N th roots of unity. We thus recover a classical result for interpolation at the roots of unity as a special case of Theorem 4.1. See [15] and the references there.

We are now ready to prove our main result about convergence of the sequence of quadrature formulas $\{I_n(f)\}$ of (4.1).

Theorem 4.2 *Suppose ρ is a complex function and ω a weight function satisfying*

$$\int_{-\pi}^{\pi} \frac{|\rho(\theta)|^2}{\omega(\theta)} d\theta < \infty$$

and let $\sum_{k=1}^{\infty} (1 - |\alpha_k|) = \infty$. Then it holds that

$$\lim_{n \rightarrow \infty} I_n(f) = I_{\rho}(f) := \int_{-\pi}^{\pi} f(e^{i\theta}) \rho(\theta) d\theta$$

for all bounded functions f on $\mathbb{T}$ for which $I_{\rho}(f)$ exists as a Riemann integral.

Proof. Let us define K by

$$\int_{-\pi}^{\pi} \frac{|\rho(\theta)|^2}{\omega(\theta)} d\theta = K^2, \quad \text{with } 0 < K < \infty, \quad (4.7)$$

and let $L_n^f(x)$ be the interpolant as in the previous theorem, then we know that $I_n(f) = I_{\rho}(L_n^f)$. The Cauchy-Schwarz inequality then gives

$$|I_{\rho}(f) - I_n(f)| = |I_{\rho}(f - L_n^f)| \leq \|f - L_n^f\|_{\omega,2} \cdot K$$

with K as in (4.7). Thus by Theorem 4.1 the proof follows. $\square$

Remark 4.4 In [6], the convergence of the sequence $\{I_n\}$ was proved when the function f is continuous on $\mathbb{T}$ and when it satisfies certain Lipschitz conditions. On the other hand, in [9], this theorem is proved for the particular case where $p(2n+1) = n$, $n = 1, 2, \dots$.

Now by Polya's Theorem [16] one has immediately

Corollary 4.3 *Let $\{A_{j,n}\}_{j=1}^n$ be the coefficients of the n -point quadrature formula $I_n(f)$. Then there exists an absolute constant $B > 0$ such that*

$$\sum_{j=1}^n |A_{j,n}| \leq B, \quad \forall n \geq 1. \quad (4.8)$$

From (4.8) and the considerations given in [6, 9], it is straightforward to deduce a result of convergence for multipoint Padé-type approximants (MPTA, for a definition see [7]) to the Riesz-Herglotz transform F_{ρ} given by (1.7). For further details about MPTAs and multipoint rational approximants (MRA) see [6, 7] and [14] for the case of the real line. We have the following theorem which should be compared with Theorem 5.1 in [9].

Theorem 4.4 *Suppose ρ is a complex function and ω a positive function satisfying (4.7) and let $\sum_{j=1}^{\infty} (1 - |\alpha_j|) = \infty$. For $n = 1, 2, \dots$ define*

$$F_n(z) = I_n(D(\cdot, z)).$$

Then

- (1) F_n is a MPTA to F_{ρ} of order $(p(n) + 1, q(n) + 1)$
- (2) $\lim_{n \rightarrow \infty} F_n(z) = F_{\rho}(z)$ uniformly on compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$.

Remark 4.5 In [6] a similar result was proved assuming the restrictive condition

$$\lim_{k \rightarrow \infty} \alpha_k = a \in \mathbb{D}$$

for the table α . Although we had in that paper also convergence results about quadrature formulas $I_n(f)$ for the case where α was included in a compact subset of $\mathbb{D}$, these results could not be used to deduce the convergence of sequences of MPTA to $F_\rho(z)$ since the sufficient condition (4.8) could not be guaranteed. This was basically due to the fact that convergence of $\{I_n(f)\}$ could not be proved in the class $C(\mathbb{T})$ of the continuous functions on $\mathbb{T}$. This drawback was overcome in the more recent paper [9], but only for the case $p(2n+1) = n$, $n = 1, 2, \dots$. Here we have the general case.

Remark 4.6 Theorem 4.1 and its consequences are still valid if we consider sequences $\{p(n)\}$ and $\{q(n)\}$ of nondecreasing nonnegative integers such that $\lim_{n \rightarrow \infty} p(n) = \lim_{n \rightarrow \infty} q(n) = \infty$. The quadrature formulas are e.g., given by

$$I_n(f) = \sum_{j=1}^{\nu(n)} A_{j,n} f(x_{j,n}), \quad \nu(n) = p(n-1) + q(n-1) + 1.$$

In our approach we have taken $\nu(n) = n$ for simplicity.

To conclude this section, we show that the property (4.8) is a direct consequence of a more general result. Our setting is as above: the nodes $\{x_{j,n}\}_{j=1}^n$ are the zeros of the para-orthogonal rational function $\tilde{\chi}_n$ with respect to the weight function ω and the table $\tilde{\alpha}$ and the quadrature formula $I_n(f) = \sum_{j=1}^n A_{j,n} f(x_{j,n})$ is exact in $\mathcal{R}_{n-1} = \mathcal{R}_{p(n-1), q(n-1)}$, i.e., it satisfies $I_n(f) = I_\rho(f)$ for all $f \in \mathcal{R}_{n-1}$. Then we prove that the bounded sequence $\{S_n\}$ with

$$S_n = \sum_{j=1}^n |A_{j,n}|$$

is convergent. Therefore, we need the orthogonal projection onto the subspace $\mathcal{R}_{(n-1)*}$. Let $\{\sigma_k\}_{k=0}^n$ be an orthonormal basis for $\mathcal{R}_n$, i.e., $\langle \sigma_j, \sigma_k \rangle = \delta_{j,k}$. Since $\sigma_{k*} \in \mathcal{R}_{k*}$ for $k = 0, 1, \dots$, it is easily seen that with $\tau_j = \sigma_{j*}$, $\{\tau_k\}_{k=0}^n$ is an orthonormal basis for $\mathcal{R}_{n*}$. Thus, for $g \in L_\omega^2(\mathbb{T})$, its orthogonal projection onto $\mathcal{R}_{n*}$ is given by

$$P_n^g = \sum_{j=0}^n c_j \tau_j, \quad c_j = \langle g, \tau_j \rangle.$$

Now setting $g(\theta) = \rho(\theta)/\omega(\theta)$ and assuming that $g \in L_\omega^2(\mathbb{T})$, we have

Theorem 4.5 Let $I_n(f) = \sum_{j=1}^n A_{j,n} f(x_{j,n})$ be an n -point quadrature formula exact in $\mathcal{R}_{n-1}$, i.e.,

$$I_n(f) = I_\rho(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \omega(\theta) d\theta, \quad \forall f \in \mathcal{R}_{n-1}$$

where the $\{x_{j,n}\}_{j=1}^n$ are the zeros of the para-orthogonal rational function $\tilde{\chi}_n$ with respect to the weight function ω and the table $\tilde{\alpha}$. If ω and ρ satisfy (4.7), then

$$A_{j,n} = w_{j,n} P_{n-1}^g(x_{j,n}), \quad j = 1, \dots, n$$

where the $\{w_{j,n}\}_{j=1}^n$ are the (positive) weights of the rational Szegő formula $\tilde{Q}_n(f)$ for the weight ω and the table $\tilde{\alpha}$.

Proof. If $L_n^f \in \mathcal{R}_{n-1}$ is the interpolant of f such that $L_n^f(x_{j,n}) = f(x_{j,n})$, $j = 1, \dots, n$, then

$$I_n(f) := \sum_{j=1}^n A_{j,n} f(x_{j,n}) = I_\rho(L_n^f).$$

If we write the Lagrange formula $L_n^f(x) = \sum_{j=1}^n \ell_{j,n}(x) f(x_{j,n})$ where $\ell_{j,n} \in \mathcal{R}_{n-1}$ and $\ell_{j,n}(x_{k,n}) = \delta_{j,k}$, $j, k = 1, \dots, n$, then

$$A_{j,n} = I_\rho(\ell_{j,n}) = \int_{-\pi}^{\pi} \ell_{j,n}(x) \frac{\rho(\theta)}{\omega(\theta)} \omega(\theta) d\theta = \int_{-\pi}^{\pi} \ell_{j,n}(x) g(\theta) \omega(\theta) d\theta, \quad x = e^{i\theta}.$$

Since $g = \rho/\omega \in L_\omega^2(\mathbb{T})$, we can consider the Fourier expansion of this function with respect to the orthonormal basis $\{\tau_j\}$ in $\mathcal{R}$, viz.

$$g(x) = \frac{\rho(x)}{\omega(x)} \sim \sum_{k=0}^{\infty} c_k \tau_k(x), \quad c_k = \langle g, \tau_k \rangle.$$

Therefore, one has by orthogonality that

$$A_{j,n} = \int_{-\pi}^{\pi} \ell_{j,n}(x) \sum_{k=0}^{n-1} c_k \tau_k(x) \omega(\theta) d\theta = \sum_{k=0}^{n-1} c_k \int_{-\pi}^{\pi} \ell_{j,n}(x) \tau_k(x) \omega(\theta) d\theta, \quad x = e^{i\theta}.$$

Now $\tau_k \in \mathcal{R}_k$ and $\ell_{j,n} \in \mathcal{R}_{n-1}$ so that by Lemma 2.4 $\ell_{j,n} \tau_k \in \tilde{\mathcal{R}}_{n-1, n-1}$. Making use of the n -point rational Szegő quadrature formula $\tilde{Q}_n(f)$, which is exact in $\tilde{\mathcal{R}}_{n-1, n-1}$, we finally have

$$A_{j,n} = \sum_{k=0}^{n-1} c_k \tilde{Q}_n(\ell_{j,n} \tau_k) = \sum_{k=0}^{n-1} c_k \tau_k(x_{j,n}) w_{j,n}$$

and this proves the theorem. $\square$

Remark 4.7 An alternative proof for Theorem 4.5 is as follows. Recall that $\tilde{Q}_n(f) = \sum_{j=1}^n w_{j,n} f(x_{j,n})$ denotes the n -point rational Szegő formula which is exact in $\tilde{\mathcal{R}}_{n-1, n-1}$ (with respect to the weight ω and the table $\tilde{\alpha}$). Thus (Lemma 2.4)

$$\tilde{Q}_n(fg_*) = I_\omega(fg_*) = \langle f, g \rangle, \quad \forall f, g \in \mathcal{R}_{n-1}.$$

Let P_{n-1}^g be the orthogonal projection of $g = \rho/\omega \in L^2(\mathbb{T})$ on $\mathcal{R}_{(n-1)*}$. We write

$$P_{n-1}^g = \sum_{j=0}^{n-1} c_j \tau_j, \quad c_j = \langle g, \tau_j \rangle, \quad j = 0, 1, \dots, n-1.$$

Recall that $\tau_j = \sigma_{j*}$ where σ_j are the orthonormal basis functions in $\mathcal{R}_{n-1}$. Thus, given $f \in \mathcal{R}_{n-1}$, we have that $\langle P_{n-1}^g, f_* \rangle = \langle g, f_* \rangle$. Indeed, $\langle P_{n-1}^g, \tau_j \rangle = \langle g, \tau_j \rangle$, because

$$\langle P_{n-1}^g, \tau_j \rangle = \left\langle \sum_{k=0}^{n-1} c_k \tau_k, \tau_j \right\rangle = c_j = \langle g, \tau_j \rangle, \quad j = 0, 1, \dots, n-1.$$

Thus for any $f \in \mathcal{R}_{n-1}$, one has

$$I_\omega(P_{n-1}^g f) = \langle P_{n-1}^g, f_* \rangle = \langle g, f_* \rangle = I_\omega(fg). \quad (4.9)$$

Now set $W_{j,n} = w_{j,n} P_{n-1}^g(x_{j,n})$ for $j = 1, \dots, n$ and define the quadrature formula $S_n(f) = \sum_{j=1}^n W_{j,n} f(x_{j,n})$, then for all $f \in \mathcal{R}_{n-1}$ we have

$$S_n(f) = \sum_{j=1}^n W_{j,n} f(x_{j,n}) = \sum_{j=1}^n W_{j,n} P_{n-1}^g(x_{j,n}) f(x_{j,n}) = \tilde{Q}_n(P_{n-1}^g f) = I_\omega(P_{n-1}^g f),$$

since $P_{n-1}^g f \in \tilde{\mathcal{R}}_{n-1,n-1}$. Therefore, by (4.9),

$$S_n(f) = I_\omega(gf) := \int_{-\pi}^{\pi} g(e^{i\theta}) f(e^{i\theta}) \omega(\theta) d\theta = \int_{-\pi}^{\pi} \rho(e^{i\theta}) f(e^{i\theta}) d\theta =: I_\rho(f).$$

Thus, $S_n(f)$ is exact in $\mathcal{R}_{n-1}$ and because such a quadrature formula is unique, it holds that $A_{j,n} = W_{j,n}$ for $j = 1, \dots, n$.

We can now state the following.

Theorem 4.6 *Suppose ρ is a complex function and ω a weight function satisfying (4.7). Assume that $\sum_{k=1}^{\infty} (1 - |\alpha_k|) = \infty$ and let $\{x_{j,n}\}_{j=1}^n$ and $\{A_{j,n}\}_{j=1}^n$ be as defined in Theorem 4.5. Then*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^n |A_{j,n}| f(x_{j,n}) = \int_{-\pi}^{\pi} f(e^{i\theta}) |\rho(\theta)| d\theta \quad (4.10)$$

for all bounded Riemann integrable functions f .

In [13], one can find the proof for interpolatory product rules based on polynomials in the interval $[-1, 1]$. In [2], Bultheel et al. gave an extension to Laurent polynomials on an interval $[a, b]$ with $0 \leq a < b < \infty$. In [9, Theorem 4.7] a proof is given for the selection $p(2n+1) = q(2n+1) = n$, $n = 1, 2, \dots$. The proof for the general case as presented above is almost the same as the proof given in [9, Theorem 4.7] and it is omitted here.

5 Rates of convergence

In this section we give estimates for the rate of convergence of a sequence of our quadrature formulas $\{I_n(f)\}$ to the integral $I_\rho(f)$ where ρ is a general complex function. To recapitulate, we consider quadrature formulas of the form

$$I_n(f) = \sum_{j=1}^n A_{j,n} f(x_{j,n}) \quad (5.1)$$

where the nodes are the zeros of paraorthogonal functions $\tilde{\chi}_n$ with respect to the positive weight ω and the table $\tilde{\alpha}$. The weights are obtained such that $I_n(f) = I_\rho(f)$ for all $f \in \mathcal{R}_{n-1} = \mathcal{R}_{p(n-1), q(n-1)}$ where $\{p(n)\}$ and $\{q(n)\}$ are two nondecreasing sequences of integers such that $p(n) + q(n) = n$. Recall that ω was just an auxiliary weight function to define the nodes. We can impose any restriction on ω to obtain whatever asymptotic properties we want the functions $\tilde{\phi}_n$ or $\tilde{\chi}_n$ to have, without jeopardising the generality of our approach (see the comments in [7]).

We are concerned with the study of the following limit

$$\limsup_{n \rightarrow \infty} |I_\rho(f) - I_n(f)|^{1/n} \quad (5.2)$$

where f is an analytic function. Now assume that α is contained in a compact subset of $\mathbb{D}$. Let $\mathcal{G}$ be the set of all regions G (closed and connected) in $\mathbb{C}$ such that $\mathbb{T}$ is in the interior of G , $0 \notin G$ and G is at a positive distance from $\{\alpha \cup \hat{\alpha}\}$ where $\hat{\alpha} = \{1/\bar{\alpha}_k : k = 1, 2, \dots\}$ and such that the boundary $\Gamma = \partial G$ is a finite union of Jordan curves. Suppose $G \in \mathcal{G}$ and f an analytic function in a neighbourhood of G . From Cauchy's and Fubini's theorems, it follows that

$$I_\rho(f) = \frac{1}{2\pi i} \int_\Gamma F_\rho(z) \left(-\frac{f(z)}{2z} \right) dz \quad (5.3)$$

with $F_\rho(z)$ the Riesz-Herglotz transform (1.7) of the complex measure $d\mu(\theta) = \rho(\theta)d\theta$.

Let $F_n(z)$ be the MPTA to $F_\rho(z)$ of order $(p(n) + 1, q(n) + 1)$ and denominator $P_n(z) = \prod_{k=1}^n (z - x_{j,n})$, i.e. $F_n(z) = (p(n)/n)_{F_\rho}(z)$. Then it holds that [7, Corollary 3.2], [6, Lemma 2.2]

$$I_\mu(f) - I_n(f) = \frac{1}{2\pi i} \int_\Gamma [F_\mu(z) - F_n(z)] \left(-\frac{f(z)}{2z} \right) dz. \quad (5.4)$$

It is clear that estimates for the limit of (5.4) follow from estimates for

$$\lim_{n \rightarrow \infty} |F_\mu(z) - F_n(z)|^{1/n} \quad (5.5)$$

uniformly on compact subsets of $\hat{\mathbb{C}} - \mathbb{T}$.

Hereafter, we assume that the sequence $\alpha = \{\alpha_k\}_1^\infty$ is included in a compact subset of $\mathbb{D}$. Then it is possible to choose the curve Γ so that $\Gamma \cap \{\alpha \cup \hat{\alpha}\} = \emptyset$.

Now we can give the main theorems of this section.

Theorem 5.1 *Assume that $\int_{-\pi}^\pi \log \omega(\theta) d\theta > -\infty$ and that α is contained in a compact subset of $\mathbb{D}$. Let F_n be the MPTA to the Riesz-Herglotz transform F_ρ whose denominator is $P_n(z) = \prod_{j=1}^n (z - x_{j,n})$ where the $\{x_{j,n}\}_{j=1}^n$ are the zeros of the para-orthogonal function $\tilde{\chi}_n$ with respect to a positive weight ω and the table $\tilde{\alpha}$. Assume ω and ρ satisfy (4.7) Then, setting $E_n = F_n - F_\rho$, we have the following estimates for the convergence of $\{F_n(z)\}$ to $F_\rho(z)$:*

(1) *For all $z \in \mathbb{D}$: $\limsup_{n \rightarrow \infty} |E_n(z)|^{1/n} \leq \exp\{r\lambda(z)\} < 1$*

(2) *For all $z \in \mathbb{E}$: $\limsup_{n \rightarrow \infty} |E_n(z)|^{1/n} \leq \exp\{s\lambda(\hat{z})\} < 1$, $\hat{z} = 1/\bar{z}$*

where $\lambda(z)$ is defined as in Lemma 2.2 and $s = 1 - r$ with $r = \lim_{n \rightarrow \infty} [p(n)/n]$.

Proof. We first consider $z \in \mathbb{D}$. Recall that $F_n(z) = Q_n(z)/P_n(z)$ with $Q_n(z) \in \Pi_n$. Because $\tilde{\chi}_n \in \tilde{\mathcal{L}}_n$, there is a constant $c_n \neq 0$ such that

$$\tilde{\chi}_n(z) = c_n P_n(z) / \tilde{\pi}_n(z). \quad (5.6)$$

Take into account that

$$\tilde{\omega}_n = \omega_{p(n)} \omega_{q(n)} \quad \text{and} \quad \tilde{\pi}_n = \pi_{p(n)} \pi_{q(n)}$$

and hence also $\tilde{B}_n = B_{p(n)} B_{q(n)}$ where $p(n) + q(n) = n$.

The error is given by (see [7, equation (3.7)])

$$E_n(z) = \frac{2z \omega_{p(n)}(z) \pi_{q(n)}(z)}{P_n(z)} \int_{-\pi}^\pi \frac{P_n(x) \rho(\theta)}{\omega_{p(n)}(x) \pi_{q(n)}(x) (x - z)} d\theta, \quad x = e^{i\theta}. \quad (5.7)$$

Note that the order of the points in the table $\tilde{A}_n$ only plays a role in the selection of the poles $x_{j,n}$, but once these are given, then the previous formula holds for any MPTA given a table containing $p + q = n$ points, whatever the order in this table is. In other words, we do not need a tilde on $\omega_{p(n)}$ and $\pi_{q(n)}$. We set $p = p(n)$ and $q = q(n)$ for simplicity of notation.

Thus from (5.6)–(5.7), one can write

$$\begin{aligned} E_n(z) &= \frac{2z\omega_p(z)}{\pi_p(z)\tilde{\chi}_n(z)} \int_{-\pi}^{\pi} \frac{\pi_p(x)\tilde{\chi}_n(x)\rho(\theta)}{\omega_p(x)(x-z)} d\theta \\ &= \frac{2zB_p(z)}{\tilde{\chi}_n(z)} \int_{-\pi}^{\pi} \frac{\tilde{\chi}_n(x)\rho(\theta)}{B_p(x)(x-z)} d\theta. \end{aligned}$$

Setting

$$M = M(z) = \max_{x \in \mathbb{T}} \int_{-\pi}^{\pi} \frac{\rho(\theta)}{|x-z|} d\theta,$$

one has

$$|E_n(z)| \leq \frac{2M|B_p(z)|}{|\tilde{\chi}_n(z)|} \|\tilde{\chi}_n\|_{\infty} \max_{x \in \mathbb{T}} |B_p^{-1}(x)|.$$

Hence (recall that $\lim_{n \rightarrow \infty} \|\tilde{\chi}_n\|_{\infty}^{1/n} = 1$)

$$\limsup_{n \rightarrow \infty} |E_n(z)|^{1/n} \leq \frac{\limsup_{n \rightarrow \infty} |B_p(z)|^{1/n}}{\liminf_{n \rightarrow \infty} |\tilde{\chi}_n(z)|^{1/n}} \limsup_{n \rightarrow \infty} [\max_{x \in \mathbb{T}} |B_p^{-1}(x)|]^{1/n}.$$

Since $|B_p(x)| = 1$ for any finite p and for any $x \in \mathbb{T}$, the last factor gives 1. Furthermore, using Theorem 2.3, we find

$$\limsup_{n \rightarrow \infty} |E_n(z)|^{1/n} \leq \frac{\lim_{n \rightarrow \infty} |B_p(z)|^{1/n}}{\lim_{n \rightarrow \infty} |\tilde{\chi}_n(z)|^{1/n}} = \frac{\lim_{n \rightarrow \infty} [|B_{p(n)}(z)|^{1/p(n)}]^{p(n)/n}}{\lim_{n \rightarrow \infty} |\tilde{\chi}_n(z)|^{1/n}}. \quad (5.8)$$

Thus, with $z \in \mathbb{D}$, it follows from (5.7), Theorem 2.3, and Lemma 2.2 that

$$\limsup_{n \rightarrow \infty} |E_n(z)|^{1/n} \leq \exp\{r\lambda(z)\}$$

Since $\text{supp}(\nu^{\alpha})$ is included in a compact subset of $\mathbb{D}$, we have for $z \in \mathbb{D}$ and $x \in \text{supp}(\nu^{\alpha})$ that the Blaschke factor satisfies

$$\left| \frac{z-x}{1-\bar{z}x} \right| < t(z) < 1.$$

Therefore $\lambda(z) < 0$ and the proof is complete for $z \in \mathbb{D}$.

Next we consider $z \in \mathbb{E}$. By Theorem 2.3 and (5.8), it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} |E_n(z)|^{1/n} &\leq \frac{\exp\{r\lambda(z)\}}{\exp\{\lambda(z)\}} = \exp\{-(1-r)\lambda(z)\} \\ &= \exp\{-s\lambda(z)\} = \exp\{s\lambda(1/\bar{z})\} < 1. \end{aligned}$$

□

Next we give an estimate for the rate of convergence of the quadrature formulas $I_n(f)$ as defined above.

Theorem 5.2 *Let ρ , ω and α and the table $\mathbb{X} = \{x_{j,n}\}$ be as in the previous theorem. Consider the quadrature formulas $I_n(f)$ with nodes given in $\mathbb{X}$ and weights such that $I_n(f) = I_\rho(f)$ for all $f \in \mathcal{R}_{n-1}$. Then it holds that*

$$\limsup_{n \rightarrow \infty} |I_\rho(f) - I_n(f)|^{1/n} \leq \gamma < 1$$

for any function f analytic in a domain $G \in \mathcal{G}$ and where $\gamma = \max\{\gamma_1, \gamma_2\}$ with

$$\gamma_1 = \max_{z \in \Gamma \cap \mathbb{D}} \exp\{r\lambda(z)\}, \quad \gamma_2 = \max_{z \in \Gamma \cap \mathbb{D}} \exp\{s\lambda(1/\bar{z})\},$$

$\lambda(z)$ is as in Lemma 2.2, $\Gamma = \partial G$ the boundary of G , $r = \lim_{n \rightarrow \infty} [p(n)/n]$ and $s = \lim_{n \rightarrow \infty} [q(n)/n]$

Proof. Make use of (5.4) and Theorem 2.3. □

Remark 5.1 Rates of convergence for quadrature formulas, both of Szegő and interpolatory type on the unit circle have been considered by the authors in previous papers. All these previous results are included in the present one. We give some examples to illustrate this.

Consider the “balanced” situation where $r = s = 1/2$. In this case, we see from Theorem 5.2 that $\gamma = \sqrt{\tilde{\gamma}}$ where $\tilde{\gamma} = \max\{\tilde{\gamma}_1, \tilde{\gamma}_2\}$ with

$$\tilde{\gamma}_1 = \max_{z \in \Gamma \cap \mathbb{D}} \exp\{\lambda(z)\} \quad \text{and} \quad \tilde{\gamma}_2 = \max_{z \in \Gamma \cap \mathbb{E}} \exp\{\lambda(1/\bar{z})\}.$$

If we assume that ρ is a positive function and we take $\alpha = \tilde{\alpha}$ and $\omega = \rho$, then the quadrature formulas given in Theorem 5.2 become rational Szegő formulas and by [7] we see that $\tilde{\gamma}$ is an estimate of the rate of convergence for such formulas. The relation $\tilde{\gamma} = \gamma^2$ was of course to be expected.

On the other hand in [6], the problem was considered under the restriction that $\lim_{n \rightarrow \infty} \alpha_n = a \in \mathbb{D}$. In that case, it holds that $\nu^\alpha = \delta_a$, so that

$$V_{\delta_a}(z) = - \int \log |z - x| d\delta_a(x) = - \log |z - a|$$

and

$$\lambda(z) = \int \log \left| \frac{z - x}{1 - \bar{z}x} \right| d\delta_a(x) = \log \left| \frac{z - a}{1 - \bar{z}a} \right|.$$

In particular, if $a = 0$, then $\lambda(z) = \log |z|$ and

$$\gamma_1 = \max_{z \in \Gamma \cap \mathbb{D}} \{|z|^{1/2}\}, \quad \gamma_2 = \max_{z \in \Gamma \cap \mathbb{E}} \{1/|z|^{1/2}\}.$$

These were the rates of convergence obtained in [6] (see also [4]).

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