# Orthogonal rational functions and quadrature on the unit circle

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**Abstract**

In this paper we shall be concerned with the problem of approximating the integral $I_\mu\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta)$, by means of the formula $I_n\{f\} = \sum_{j=1}^{n} A_j^{(n)} f(x_j^{(n)})$ where $\mu$ is some finite positive measure. We want the approximation to be so that $I_n\{f\} = I_\mu\{f\}$ for $f$ belonging to certain classes of rational functions with prescribed poles which generalize in a certain sense the space of polynomials. In order to get nodes $\{x_j^{(n)}\}$ of modulus 1 and positive weights $A_j^{(n)}$, it will be fundamental to use rational functions orthogonal on the unit circle analogous to Szegő polynomials.

## 1 Introduction

In this paper, we are concerned with complex function theory on the unit circle. We start with the introduction of some notation for the unit circle, the open unit disc and the exterior of the unit circle

$$\mathbf{T} = \{z : |z| = 1\}; \quad \mathbf{D} = \{z : |z| < 1\}; \quad \mathbf{E} = \{z : |z| > 1\}.$$

For a given sequence $\{\alpha_i\}_{i=1}^{\infty} \subset \mathbf{D}$, we consider for $n = 0, 1, \ldots$ the nested spaces $\mathcal{L}_n$ of rational functions of degree $n$ at most which are spanned by the basis of partial Blaschke products $\{B_k\}_{k=0}^{n}$ where $B_0 = 1$; $B_n = B_{n-1}\zeta_n$ for $n = 1, 2, \ldots$ and the Blaschke factors are defined as

$$\zeta_n(z) = \frac{\overline{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \overline{\alpha}_n z}$$

By convention, we set $\overline{\alpha}_n/|\alpha_n| = -1$ for $\alpha_n = 0$. Sometimes, we shall also write

$$B_n(z) = \eta_n \frac{\omega_n(z)}{\pi_n(z)}; \quad \eta_n = (-1)^n \prod_{j=1}^{n} \frac{\overline{\alpha}_j}{|\alpha_j|}; \quad \omega_n(z) = \prod_{j=1}^{n}(z - \alpha_j) \text{ and } \pi_n(z) = \prod_{j=1}^{n}(1 - \overline{\alpha}_j z). \tag{1.1}$$

These spaces of rationals have been studied in connection with the Pick-Nevanlinna problem [9, 10, 8] and in many applications [1, 2, 5]. Note that if all the $\alpha_i$ are equal to zero, the spaces $\mathcal{L}_n$ collapse to the spaces $\Pi_n$ of polynomials of degree $n$. Clearly $\mathcal{L}_n$ is a space of rational functions with prescribed poles $1/\overline{\alpha}_i$; $i = 1, 2, \ldots, n$ which are all in $\mathbf{E}$, that is,

$$\mathcal{L}_n = \text{span}\{B_k; k = 0, 1, \ldots, n\} = \{\frac{p_n}{\pi_n}; p_n \in \Pi_n\}$$

We also introduce the following transformation $f_*(z) = \overline{f(1/\overline{z})}$ ($f_*(z) = \overline{f(z)}$ on $\mathbf{T}$), which allows to define for $f_n \in \mathcal{L}_n$ the superstar conjugate as

$$f_n^*(z) = B_n(z) f_{n*}(z).$$

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Note that in the polynomial case, i.e., when $\alpha_i = 0$; $i = 0, \ldots, n$, hence $B_n(z) = z^n$, the coefficient $\overline{p^*(0)}$ is the leading coefficient. In analogy, we shall call here $\overline{f_n^*(\alpha_n)}$ the leading coefficient of $f_n \in \mathcal{L}_n$ (with respect to the basis $\{B_k\}$).

Next we consider a positive measure $\mu$ on the unit circle and orthonormalize the basis for $\mathcal{L}_n$ with respect to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta)$$

to generate an orthonormal system $\phi_0 = 1$, $\phi_k \in \mathcal{L}_k - \mathcal{L}_{k-1}$, $\phi_k \perp \mathcal{L}_{k-1}$, $k = 1, 2, \ldots, n$. They are uniquely defined if we require that their leading coefficient $\kappa_n = \phi_n^*(\alpha_n)$ is positive. In [2] it was proved that all the zeros of $\phi_n^*(z)$ are in $\mathbf{E}$ and that it satisfies the orthogonality property $\phi_n^* \perp \zeta_n \mathcal{L}_{n-1}$, $n = 1, 2, \ldots$ where

$$\zeta_n \mathcal{L}_{n-1} = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\} = \mathcal{L}_n(\alpha_n). \tag{1.2}$$

(Obviously, the zeros of $\phi_n$ lie in $\mathbf{D}$).

Other bases can be used for $\mathcal{L}_n$ (See [2, 6]). In particular, the following will be of great interest for our purposes. If we set $V_0 = 1$, $V_k = B_{k-1}/(1 - \overline{\alpha}_k z)$; $k = 1, 2, \ldots$, then in [2] it was proved that $\mathcal{L}_n = \text{span}\{1, (z-w)V_1, (z-w)V_2, \ldots, (z-w)V_n\}$, $w$ being any complex number.

In this paper we shall study how to obtain quadrature formulas of the form

$$I_n\{f\} = \sum_{j=1}^{n} A_j^{(n)} f(x_j^{(n)}); \quad x_i^{(n)} \neq x_j^{(n)}, i \neq j; \quad x_j^{(n)} \in \mathbf{T}, j = 1, \ldots, n \tag{1.3}$$

(sometimes we shall drop the superscripts if no confusion is possible). It approximates the integral

$$I_\mu\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta) = \int f(z)d\mu(z). \tag{1.4}$$

It is well known that such quadrature formulas are of great interest in the polynomial case to solve the trigonometric moment problem or equivalently the Schur coefficient problem. The same can be said for the rational case where the Schur coefficient problem is generalized to the Pick-Nevanlinna interpolation problem. We shall generalize the basic ideas of the quadrature formulas on the real line where the use of interpolating functions (which are easily integrated) and the zeros of orthogonal functions become extremely important.

Our interpolating function spaces will be of the form

$$\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q = \{\frac{P}{\omega_p \pi_q}; P \in \Pi_{p+q}\}$$

($p$ and $q$ are nonnegative integers, $\omega_p$ and $\pi_q$ as in (1.1)). Observe that $\mathcal{L}_{n*} = \text{span}\{1, B_{1*}, \ldots, B_{n*}\} = \{1, 1/B_1, \ldots, 1/B_n\}$. Therefore,

$$\mathcal{R}_{p,q} = \text{span}\{\frac{1}{B_p}, \frac{1}{B_{p-1}}, \ldots, \frac{1}{B_1}, 1, B_1, \ldots, B_q\}$$

($\mathcal{R}_{0,n} = \mathcal{L}_n$). When all the $\alpha_i$ are equal to zero, then $B_k = z^k$ and one has $\mathcal{R}_{p,q} = \text{span}\{z^k : k = -p, \ldots, q\} = \Delta_{-p,q}$, that is the space of Laurent polynomials, or functions of the form $\Delta_{-p,q} = \{L(z) = \sum_{j=-p}^{q} \gamma_i z^i; \gamma_i \in \mathbf{C}\}$

## 2 Interpolatory quadrature formulas

Writing $A = \{\alpha_i\}_{i=1}^{\infty}$ and $\hat{A} = \{1/\overline{\alpha}_i : \alpha_i \in A\}$, it is easily seen [4] that $\mathcal{R}_{p,q}$ represents a Chebyshev system on any set $X \subset \mathbf{C} - (A \cup \hat{A})$, and the following holds

**Proposition 1** *Let the distinct numbers $\{x_j\}_{j=1}^{n} \subset \mathbf{C} - (A \cup \hat{A})$ be given. Then, for arbitrary $\{w_j\}_{j=1}^{n} \subset \mathbf{C}$, there exists a unique $p_n \in \mathcal{R}_{p,q}$ $(p + q = n - 1)$ such that $p_n(x_j) = w_j$, $j = 1, \ldots, n$.*

For our purposes, it will be convenient to give Lagrange type formulas for $p_n$. Therefore set

$$\Omega_n(z) = \frac{N_n(z)}{\omega_p(z)\pi_{q+1}(z)} \in \mathcal{R}_{p,q+1} \tag{2.1}$$

where $N_n(z) = \prod_{j=1}^{n}(z - z_j)$ is the *node polynomial*, and define

$$L_{j,n}(z) = \frac{1-\overline{\alpha}_{q+1}z}{1-\overline{\alpha}_{q+1}x_j}\,\frac{\Omega_n(z)}{(z-x_j)\Omega_n'(x_j)} \in \mathcal{R}_{p,q}, \quad j = 1,2,\dots,n \tag{2.2}$$

Then it can be checked that

$$p_n(z) = \sum_{j=1}^{n} L_{i,n}(z)w_j. \tag{2.3}$$

Indeed, observe that

$$L_{i,n}(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad 1 \leq i,j \leq n.$$

In particular if $p = 0$ $(q = n-1)$, then $\mathcal{R}_{0,q} = \mathcal{L}_{n-1}$ and the interpolant will be given by

$$p_n(z) = \sum_{j=1}^{n} \frac{1-\overline{\alpha}_n z}{1-\overline{\alpha}_n x_j}\,\frac{\Omega_n(z)}{(z-x_j)\Omega_n'(x_j)} w_j$$

where $\Omega_n(z) = N_n(z)/\pi_n(z)$. When all the $\alpha_i$ are equal to zero, then $\Omega_n(z) = N_n(z)$ and the well known Lagrange interpolation formula for polynomial interpolation is recovered.

The space $\mathcal{R}_{p,q}$ will be called *a domain of validity* for the formula $I_n\{f\}$ given by (1.3) if

$$E_n\{f\} = I_\mu\{f\} - I_n\{f\} = 0, \quad \forall f \in \mathcal{R}_{p,q}.$$

Furthermore, $\mathcal{R}_{p,q}$ is said to be *maximal domain of validity* when $p = q$ and neither $\mathcal{R}_{p+1,q}$ nor $\mathcal{R}_{p,q+1}$ is a domain of validity. In [7] (see also [3]) it was proved for the polynomial case that $\Delta_{-(n-1),n-1}$ is a maximal domain of validity. Here we shall prove a similar result for the rational case, that is we prove that $\mathcal{R}_{n-1,n-1}$ is a maximal domain of validity.

First let us assume that the distinct nodes $\{x_j\}_{j=1}^{n} \subset \mathbf{T}$ are given. Then, since $\mathcal{R}_{p,q}$ with $p$ and $q$ nonnegative integers $(p + q = n-1)$ represents a Chebyshev system on $\mathbf{T}$, weights $A_1,\dots,A_n$ can be uniquely determined so that the formula $I_n\{f\} = \sum_{j=1}^{n} A_j f(x_j)$ has domain of validity $\mathcal{R}_{p,q}$ at least.

On the other hand, for a function $f$, defined on $\mathbf{T}$, let $p_n \in \mathcal{R}_{p,q}$ be its interpolant in the nodes $\{x_j\}_{j=1}^{n}$, then

$$\int p_n(z)d\mu(z) = \sum_{j=1}^{n}[\int L_{j,n}(z)d\mu(z)]f(x_j) = \sum_{j=1}^{n} A_j f(x_j) \tag{2.4}$$

Such a quadrature formula is called of *interpolatory type* in $\mathcal{R}_{p,q}$. The following proposition is readily proved.

**Proposition 2** *A formula $I_n\{f\} = \sum_{j=1}^{n} A_j f(x_j)$ with distinct nodes $\{x_j\}_{j=1}^{n} \subset \mathbf{T}$ has a domain of validity $\mathcal{R}_{p,q}$ $(p+q = n-1)$ if and only if it is of interpolatory type in that subspace.*

Finally, we have the following result concerning the maximality of the domain of validity:

**Theorem 1** *For each $n \geq 1$, there can not exist an n-point quadrature formula of the form (1.3) with distinct nodes on $\mathbf{T}$ and with a domain of validity $\mathcal{R}_{n-1,n}$ or $\mathcal{R}_{n,n-1}$.*

**Proof.** Let $n \geq 1$ be given. First we assume that there exists an $n$-point formula (1.3) that is valid for all $f \in \mathcal{R}_{n-1,n}$. Let $N_n(z) = \prod_{j=1}^{n}(z - x_j) \in \Pi_n$ be the node polynomial and introduce $R_n(z) = \lambda_n N_n(z)/\pi_n(z) \in \mathcal{L}_n$, $\lambda_n \neq 0$. The parameter $\lambda_n$ is chosen so that the leading coefficient of $R_n(z)$ is positive $(R_n^*(\alpha_n) > 0)$ and $\langle R_n, R_n\rangle = 1$. On the other hand, $R_n(z)/B_k(z) \in \mathcal{R}_{n-1,n}$ for $0 \leq k \leq n-1$ and therefore

$$\langle R_n, B_k\rangle = \int R_n(z)\overline{B_k(z)}d\mu(z) = \int R_n(z)B_{k*}(z)d\mu = \int \frac{R_n(z)}{B_k(z)}d\mu = \sum_{j=1}^{n} A_j \frac{R_n(x_j)}{B_k(x_j)} = 0$$

(Recall that $R_n(x_j) = 0$, $j = 1, 2, \ldots, n$). Similarly, let us assume that $I_n\{f\} = I_\mu\{f\}$ for all $f \in \mathcal{R}_{n,n-1}$. Since

$$R_{n*}(z) = \overline{\lambda}_n \frac{N_{n*}(z)}{\pi_{n*}(z)} = \gamma_n \frac{N_n(z)}{\omega_n(z)}; \quad \gamma_n \neq 0;$$

also $R_{n*}(x_j) = 0$, $j = 1, 2, \ldots, n$. Because $B_k R_{n*} \in \mathcal{R}_{n,n-1}$ for $0 \leq k \leq n-1$ we also get

$$\langle B_k, R_n \rangle = \int B_k(z)\overline{R_n(z)} d\mu = \int B_k(z) R_{n*}(z) d\mu = \sum_{j=1}^{n} A_j B_k(x_j) R_{n*}(x_j) = 0$$

Thus we have found $R_n \in \mathcal{L}_n - \mathcal{L}_{n-1}$, $R_n(x_j) = 0$, $j = 1, \ldots, n$, $x_j \in \mathbf{T}$, which has a positive leading coefficient and satisfies in both cases

$$\langle R_n, B_k \rangle = 0; \quad 0 \leq k \leq n-1; \text{ and } \langle R_n, R_n \rangle = 1.$$

It follows that $R_n = \phi_n$. However, this is impossible since the zeros of $\phi_n$ lie outside the closed unit disk and not on $\mathbf{T}$. □

**Remark:** This theorem says that if an $n$-point formula $I_n\{f\}$ (1.3) with nodes on $\mathbf{T}$ exists so that $I_n\{f\} = I_\mu\{f\}$ for all $f \in \mathcal{R}_{n-1,n-1}$, then this subspace is a maximal domain of validity.

# 3 Para-orthogonal rational functions

As we have mentioned above, the zeros of the $n$-th orthonormal function $\phi_n$ lie all in $\mathbf{D}$. In order to develop quadrature formulas on $\mathbf{T}$, it is useful to have functions in $\mathcal{L}_n$ with orthogonality properties with respect to $\langle \cdot, \cdot \rangle$ whose zeros are distinct and lie on $\mathbf{T}$. For that purpose, we consider now sequences of rational functions in $\mathcal{L}_n$ which we shall call *para-orthogonal* (we follow the terminology of the polynomial case – see [7]) because of deficiencies in their orthogonality properties. A sequence $\{\chi_n \in \mathcal{L}_n, n = 0, 1, 2 \ldots\}$ is said to be para-orthogonal if it satisfies ($\mathcal{L}_n(\alpha_n)$ is defined in (1.2))

(i) $\langle \chi_n, 1 \rangle \neq 0$; $\langle \chi_n, B_n \rangle \neq 0$, $n \geq 0$

(ii) $\chi_n \perp \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, $n \geq 1$

Clearly, the sequence of orthonormal functions $\{\phi_n\}$ is not para-orthogonal ($\langle \phi_n, 1 \rangle = 0$), neither is the sequence $\{\phi_n^*\}$ ($\langle \phi_n^*, B_n \rangle = \langle 1, \phi_n \rangle = 0$). However, we can obtain a para-orthogonal sequence by considering functions of the form

$$f_n(z; w_n) = \phi_n(z) + w_n \phi_n^*(z), \quad n = 0, 1, \ldots \tag{3.1}$$

Properties of these functions are described in the following theorem.

**Theorem 2** *Let $f_n(z; w_n)$ be defined as in (3.1) above.*
*(1) Let $c_n, w_n$ be given nonzero complex numbers. Then $\{c_n f_n(z; w_n)\}$ is a para-orthogonal sequence.*
*(2) Let $\{\chi_n\}$ be a para-orthogonal sequence. Then for $n \geq 0$, there exist nonzero complex numbers $c_n, d_n$ such that*

$$\chi_n(z) = c_n f_n(z; w_n); \quad w_n = d_n / c_n.$$

**Proof.** (i) The case $n = 0$ is trivial, so we suppose $n \geq 1$. If $f_n(z; w_n) = \phi_n(z) + w_n \phi_n^*(z)$, $w_n \neq 0$, then $\langle f_n, 1 \rangle = \langle \phi_n, 1 \rangle + w_n \langle \phi_n^*, 1 \rangle = w_n \langle \phi_n^*, 1 \rangle$ but $\langle \phi_n^*, 1 \rangle = \int \phi_n^* d\mu = \int B_n \phi_n^* d\mu = \langle B_n, \phi_n \rangle \neq 0$. Similarly $\langle f_n, B_n \rangle \neq 0$. Since $\phi_n \perp \mathcal{L}_{n-1}$ and $\phi_n^* \perp \mathcal{L}_n(\alpha_n)$, we clearly have

$$\langle f_n, f \rangle = 0, \quad \forall f \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n).$$

(ii) Let us first determine $c_n$ and $d_n$ by imposing that $T_n = \chi_n - c_n \phi_n - d_n \phi_n^*$ satisfies

$$\langle T_n, 1 \rangle = 0 \text{ and } \langle T_n, \phi_n \rangle = 0. \tag{3.2}$$

This gives $\langle T_n, 1 \rangle = \langle \chi_n - c_n \phi_n - d_n \phi_n^*, 1 \rangle = \langle \chi_n, 1 \rangle - d_n \langle \phi_n^*, 1 \rangle$, where $\langle \phi_n^*, 1 \rangle \neq 0$. Hence $d_n = \langle \chi_n, 1 \rangle / \langle \phi_n^*, 1 \rangle$. Similarly $\langle T_n, \phi_n \rangle = \langle \chi_n, \phi_n \rangle - c_n - d_n \langle \phi_n^*, \phi_n \rangle = 0$ which yields $c_n = \langle \chi_n, \phi_n \rangle - \langle \chi_n, 1 \rangle \langle \phi_n^*, \phi_n \rangle / \langle \phi_n^*, 1 \rangle$.

Next we show that $T_n \equiv 0$ for $n \geq 0$. It is immediate that $T_0 = 0$. For $n \geq 1$, we can express $T_n$ in the form $T_n(z) = \sum_{k=1}^{n} a_k \phi_k(z)$, $a_k \in \mathbf{C}$. By (3.2) we obtain that $0 = \langle T_n, 1 \rangle = a_0 \langle \phi_0, 1 \rangle = a_0$.

Similarly, $0 = \langle T_n, \phi_n \rangle = a_n \langle \phi_n, \phi_n \rangle = a_n$. Therefore, in particular $T_1 \equiv 0$. We now use the basis $U_0 = 1$, $U_k = (z-w)V_k$, $k = 1, 2, \ldots$, $w \in \mathbf{C}$, that is

$$U_0 = 1, \quad U_k = \frac{(z-w)B_{k-1}}{1-\overline{\alpha}_k z}, \quad k = 1, 2, \ldots \tag{3.3}$$

and choose $w = \alpha_n$. Observe then that $U_k \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, for $k = 1, 2, \ldots, n-1$. Therefore, for $n \geq 2$

$$0 = \langle T_n, U_1 \rangle = \sum_{k=1}^{n-1} a_k \langle \phi_k, U_1 \rangle = a_1 \langle \phi_1, U_1 \rangle$$

because $\chi_n$ is para-orthogonal. This implies $a_1 = 0$. Continuing in this manner for $U_k$, $k = 2, \ldots, n-1$, we can show that $a_2 = a_3 = \cdots = a_{n-1} = 0$. Thus, $T_n \equiv 0$ for $n \geq 0$ and thus

$$\chi_n = c_n \phi_n + d_n \phi_n^*, \quad n = 0, 1, \ldots$$

Obviously, $d_n \neq 0$ ($\langle \chi_n, 1 \rangle \neq 0$). If $c_n = 0$, then $\chi_n = d_n \phi_n^*$ and $d_n \langle \phi_n^*, B_n \rangle = \langle \chi_n, B_n \rangle \neq 0$ which is a contradiction. In conclusion, we can then write for all $n \geq 0$

$$\chi_n(z) = c_n(\phi_n(z) + \frac{d_n}{c_n}\phi_n^*(z)) = c_n f_n(z; w_n); \quad w_n = \frac{d_n}{c_n}.$$

□

Let $\chi_n \in \mathcal{L}_n$ be given. For $k \in \mathbf{C}$, $k \neq 0$, $\chi_n$ is called *$k$-invariant* if

$$\chi_n^*(z) = k\chi_n(z) \text{ for all } z \in \mathbf{C}.$$

A sequence $\{\chi_n \in \mathcal{L}_n : n = 0, 1, \ldots\}$ is said to be *$\{k_n\}$-invariant* if $\chi_n$ is $k_n$-invariant. Note that $\phi_n$ is not $k$-invariant for any $k$ because $\phi_n^* = k\phi_n$ would imply that $\langle \phi_n^*, 1 \rangle = k\langle \phi_n, 1 \rangle = 0$ which is impossible because $\langle \phi_n^*, 1 \rangle = \langle B_n, \phi_n \rangle \neq 0$.

For a $\{k_n\}$-invariant sequence $\{\chi_n\}$ we can prove a theorem that is similar to theorem 2.

**Theorem 3** *Let $f_n(z; w_n)$ be as defined in (3.1).*

*(i) For all $n \geq 0$, let $c_n, w_n \in \mathbf{C}$ be given ($c_n \neq 0$, $|w_n| = 1$) and let $k_n = \overline{c}_n \overline{w}_n / c_n$. Then $\{c_n f_n(z; w_n)\}$ is a $\{k_n\}$-invariant sequence.*

*(ii) Let $\{\chi_n \in \mathcal{L}_n, n = 0, 1, \ldots\}$ be a para-orthogonal and $\{k_n\}$-invariant sequence, then $\chi_n(z) = c_n f_n(z; w_n)$ for all $z \in \mathbf{C}$ such that $|w_n| = 1$ and $k_n = \overline{c}_n \overline{w}_n / c_n \in \mathbf{T}$.*

**Proof.** (i) It suffices to note that

$$[c_n f_n(z; w_n)]^* = \overline{c}_n[\phi_n^* + \overline{w}_n \phi_n] = \frac{\overline{c}_n}{\overline{w}_n}[\phi_n + w_n \phi_n^*] = \frac{\overline{c}_n \overline{w}_n}{c_n} c_n f_n(z; w_n).$$

(ii) By virtue of theorem 2 we can write

$$\chi_n = c_n \phi_n + d_n \phi_n^* = c_n(\phi_n + w_n \phi_n^*); \quad w_n = c_n / d_n$$

Now, suppose that for some $n \geq 0$, $\chi_n$ is $k_n$-invariant, that is $\chi_n^* = k_n \chi_n$. Then $\overline{c}_n \phi_n^* + d_n \phi_n = k_n(c_n \phi_n + d_n \phi_n^*)$ or equivalently $(\overline{d}_n - k_n c_n)\phi_n + (\overline{c}_n - k_n d_n)\phi_n^* = 0$. Since $\phi_n$ and $\phi_n^*$ are linearly independent, we can conclude that $k_n = \overline{d}_n / c_n = \overline{c}_n / d_n$. This implies $|c_n| = |d_n|$ and hence $|w_n| = 1$. □

Now we can prove a theorem which will provide the nodes for the desired $n$-point quadrature formula.

**Theorem 4** *Let $\{\chi_n \in \mathcal{L}_n, n = 0, 1, \ldots\}$ be a $\{k_n\}$-invariant and para-orthogonal sequence. Then, for all $n \geq 1$, $\chi_n$ has $n$ simple zeros which lie on the unit circle $\mathbf{T}$.*

**Proof.** Let be $n \geq 1$, then $\chi_n = c_n[\phi_n + w_n \phi_n^*]$, $c_n \neq 0$, $|w_n| = 1$. Therefore, it suffices to prove that a function of the form $f_n(z) = \phi_n(z) + w\phi_n^*(z)$, ($w \in \mathbf{T}$) has exactly $n$ simple zeros on $\mathbf{T}$. This is proved in [2]. □

# 4 Rational Szegő formulas

For given $n \geq 1$, let $x_1, \ldots, x_n$ be the zeros of $f_n(z) = \phi_n(z) + w\phi_n^*(z)$, $(|w| = 1)$. We shall see how these zeros can be used to construct an $n$-point formula $I_n\{f\}$ of the form (1.3).

**Theorem 5** *Let $x_1, \ldots, x_n$ be the zeros of $f_n(z; w)$, $(|w| = 1)$. Then there exist positive $A_1, \ldots, A_n$ such that the formula*

$$I_n\{f\} = \sum_{j=1}^{n} A_j f(x_j) \tag{4.1}$$

*is exact ($I_n\{f\} = I_\mu\{f\}$) for all $f \in \mathcal{R}_{n-1,n-1}$.*

**Proof.** See [2]. □

From this theorem and theorem 1, we conclude that $\mathcal{R}_{n-1,n-1}$ is a maximal domain of validity.

The next result says that the only quadrature formulas with such a maximal domain of validity are precisely those given in theorem 5.

**Theorem 6** *Let us consider the $n$-point formula as in (4.1) with $x_i \neq x_j, |x_i| = 1$. Then $I_n\{f\}$ has a domain of validity $\mathcal{R}_{n-1,n-1}$ if and only if*

*(i) $I_n\{f\}$ is of interpolatory type in $\mathcal{R}_{p,q}$, $p$ and $q$ being nonnegative integers such that $p + q = n - 1$.*

*(ii) If we write $\chi_n(z) = N_n(z)/\pi_n(z)$, where $N_n(z) = \prod_{j=1}^{n}(z - x_j)$ is the node polynomial, then $\chi_n$ is para-orthogonal and $k$-invariant.*

**Proof.** "$\Rightarrow$" (i) Let $p$ and $q$ satisfy $p + q = n - 1$. Clearly $\mathcal{R}_{p,q} \subset \mathcal{R}_{n-1,n-1}$. Since $I_n\{f\}$ is exact in $\mathcal{R}_{p,q}$, it must be of interpolatory type by proposition 2.

(ii) Let us first prove that $\chi_n$ is $k$-invariant.

$$\chi_n^*(z) = B_n(z)\chi_{n*}(z) = B_n(z)\prod_{j=1}^{n}\frac{\overline{x}_j(z - x_j)}{\alpha_j - z} = k_n\frac{N_n(z)}{\pi_n(z)} = k_n\chi_n(z)$$

with $k_n = \prod_{j=1}^{n}(\overline{\alpha}_j\overline{x}_j/|\alpha_j|) \in \mathbf{T}$.

To prove para-orthogonality, assume $0 = \langle\chi_n, 1\rangle = \int\chi_n(z)d\mu(z) = I_n\{\chi_n\}$. But $\chi_n \in \mathcal{L}_n - \mathcal{L}_{n-1}$. Now $\mathcal{R}_{n-1,n} = \mathcal{R}_{n-1,n-1} \cup (\mathcal{L}_n - \mathcal{L}_{n-1})$. So, $I_n\{f\}$ is exact in $\mathcal{R}_{n-1,n}$ which contradicts theorem 1. Thus $\langle\chi_n, 1\rangle \neq 0$. On the other hand, also $\langle B_n, \chi_n\rangle = \int B_n\chi_{n*}d\mu = \langle\chi_n^*, 1\rangle = k_n\langle\chi_n, 1\rangle \neq 0$.

Finally, we must show that $\langle\chi_n, f\rangle = 0$ for all $f \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$. Let us again consider the basis $\{U_0, U_1, \ldots, U_n\}$ defined in (3.3) where we replace $w$ by $\alpha_n$. If $f \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$, then $f(z) = \sum_{k=0}^{n-1} a_k U_k(z)$, with $a_0 = 0$ since $f(\alpha_n) = 0$. Consequently, we have to prove that $\langle\chi_n, U_k\rangle = 0$, $1 \leq k \leq n-1$, $(n \geq 2)$. Now, $U_{k*}(z) = \lambda_k(1 - \overline{\alpha}_n z)\pi_{k-1}(z)/\omega_k(z)$, $\lambda_k \neq 0$. Thus

$$\begin{aligned}\langle\chi_n, U_k\rangle &= \int\chi_n(z)U_{k*}(z)d\mu = \lambda_k\int\frac{N_n(z)(1 - \overline{\alpha}_n z)\pi_{k-1}(z)}{\pi_n(z)\omega_k(z)}d\mu \\ &= \lambda_k\int g_{n,k}(z)d\mu = \lambda_k I_n\{g_{n,k}\} = 0\end{aligned}$$

since $g_{n,k} \in \mathcal{R}_{n-1,n-1}$ and $g_{n,k}(x_j) = 0$, $j = 1, 2, \ldots, n$.

"$\Leftarrow$" Let $\chi_n \in \mathcal{L}_n$ be para-orthogonal and $k$-invariant. $\chi_n$ has $n$ simple zeros $\{x_j\}_{j=1}^{n}$ on $\mathbf{T}$. Let $p$ and $q$ be nonnegative integers such that $p + q = n - 1$. Consider $L_j^{(p)}(z) \in \mathcal{R}_{p,q}$ defined by $L_j^{(p)}(x_i) = \delta_{ij}$, $1 \leq i, j \leq n$ and set $A_j^{(p)} = \int L_j^{(p)}(z)d\mu$. We prove that $I_n\{f\} = \sum_{j=1}^{n} A_j^{(p)} f(x_j)$ has domain of validity $\mathcal{R}_{n-1,n-1}$. Therefore suppose $T \in \mathcal{R}_{n-1,n-1}$ and define $R(z) = T(z) - \sum_{j=1}^{n} L_j^{(p)}(z)R(x_j)$. Clearly $R \in \mathcal{R}_{n-1,n-1}$ and $R(x_j) = 0$, $j = 1, 2, \ldots, n$. So

$$R(z) = \frac{P(z)}{\pi_{n-1}(z)\omega_{n-1}(z)}; \quad P \in \Pi_{2n-2} \text{ and } P(x_j) = 0.$$

By the latter condition, $P(z)$ is of the form $P(z) = S(z)N_n(z)$ where $N_n(z) = \prod_{j=1}^{n}(z - x_j)$ is the node polynomial and $S \in \Pi_{n-2}$. Thus $R(z)$ can be brought into the form

$$R(z) = \frac{N_n(z)}{\pi_n(z)}\frac{(1 - \overline{\alpha}_n z)S(z)}{\omega_{n-1}(z)} = \chi_n(z)g_{n*}(z) \text{ where } g_n(z) = \frac{(z - \alpha_n)S^*(z)}{\pi_{n-1}(z)} \in \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n).$$

Hence $\int R(z)d\mu = \int \chi_n(z)\overline{g_n(z)}d\mu = \langle \chi_n, g_n \rangle = 0$ due to the para-orthogonality of $\chi_n$. Thus $I_\mu\{T\} = I_n\{T\}$ for all $T \in \mathcal{R}_{n-1,n-1}$.

It remains to be shown that the weights are independent of $p$. Given $p$ and $p'$, $0 \le p, p' \le n-1$, one has to prove that $A_j^{(p)} = A_j^{(p')}$, $j = 1, \ldots, n$. Let us assume that $p < p'$, that is $p' = p + r$, $1 \le r \le n-1$. Clearly it suffices to take $r = 1$. So suppose $p' = p + 1$. One has $A_j^{(p)} = \int L_j^{(p)}(z)d\mu$, where

$$L_j^{(p)}(z) = \frac{1-\overline{\alpha}_{q+1}z}{1-\overline{\alpha}_{q+1}x_j}\frac{\Omega_{n,p}(z)}{(z-x_j)\Omega'_{n,p}(x_j)} \text{ and } \Omega_{n,p}(z) = \frac{N_n(z)}{\omega_p(z)\pi_{q+1}(z)}$$

For $p' = p+1$, one has

$$\Omega_{n,p+1}(z) = \frac{N_n(z)}{\omega_{p+1}(z)\pi_q(z)} = \frac{1-\overline{\alpha}_{q+1}z}{z-\alpha_{p+1}}\Omega_{n,p}(z)$$

and

$$\Omega'_{n,p+1}(x_j) = \frac{1-\overline{\alpha}_{q+1}x_j}{x_j-\alpha_{p+1}}\Omega'_{n,p}(x_j).$$

Therefore,

$$L_j^{(p+1)}(z) = \frac{1-\overline{\alpha}_q z}{1-\overline{\alpha}_q x_j}\frac{\Omega_{n,p+1}(z)}{(z-x_j)\Omega'_{n,p+1}(x_j)} = \frac{1-\overline{\alpha}_q z}{1-\overline{\alpha}_q x_j}\frac{x_j-\alpha_{p+1}}{z-\alpha_{p+1}}L_j^{(p)}(z)$$

or equivalently,

$$L_j^{(p+1)}(z) = L_j^{(p)}(z) + \frac{(z-x_j)(\alpha_p - \overline{\alpha}_q)}{(z-\alpha_{p+1})(1-\overline{\alpha}_q x_j)}L_j^{(p)}.$$

Hence, one has to prove that

$$\int \frac{z-x_j}{z-\alpha_{p+1}}L_j^{(p)}(z)d\mu = 0 \text{ or equivalently } \int \frac{1-\overline{\alpha}_{q+1}z}{z-\alpha_{p+1}}\Omega_{n,p}(z)d\mu = 0.$$

The latter integral can be written in the form

$$\int \frac{N_n(z)}{\pi_q(z)\omega_{p+1}(z)}d\mu = \int \frac{N_n(z)}{\pi_n(z)}\frac{(1-\overline{\alpha}_{q+1}z)\cdots(1-\overline{\alpha}_n z)}{\omega_{p+1}(z)}d\mu = \int \chi_n(z)h_*(z)d\mu$$

where $h(z) = (z-\alpha_{q+1})\cdots(z-\alpha_n)/\pi_{p+1}(z)$. Observe that $p' = p+1 \le n-1$ and consequently $h \in \mathcal{L}_{n-1}$ as well as $h \in \mathcal{L}_n(\alpha_n)$. By para-orthogonality of $\chi_n$ we may conclude that $\int \chi_n h_* d\mu = \langle \chi_n, h \rangle = 0$. □

As a result, we have now a one-parameter family of quadrature formulas of the form (4.1), depending on a parameter $w \in \mathbf{T}$, such that

(i) the nodes are the zeros of $\phi_n + w\phi_n^*$ and

(ii) the weights are given by $A_j^{(n)} = \int L_{j,n}(z)d\mu(z)$ where $L_{j,n} \in \mathcal{R}_{0,n-1} = \mathcal{L}_{n-1}$ is defined by $L_{j,n}(x_i^{(n)}) = \delta_{ij}$ and

(iii) $\mathcal{R}_{n-1,n-1}$ is a maximal domain of validity.

Such a formula will be called an $n$-point rational Szegő quadrature formula, or an R-Szegő quadrature for short.

# References


[1] A. Bultheel and P. Dewilde. Orthogonal functions related to the Nevanlinna-Pick problem. In P. Dewilde, editor, *Mathematical Theory of Networks and Systems*, pages 207–211, North Hollywood, 1979. Western Periodicals. Proceedings MTNS Conference, Delft, The Netherlands.

[2] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. A Szegő theory for rational functions. Technical Report TW-131, K. U. Leuven, Dept. of Computer Science, May 1990.

[3] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. Orthogonality and quadrature on the unit circle. In C. Brezinski, L. Gori, and A. Ronveaux, editors, *Orthogonal polynomials and their applications*, volume 9 of *IMACS annals on computing and applied mathematics*, pages 205–210, Basel, 1991. J.C. Baltzer AG.

[4] P.J. Davis. *Interpolation and approximation.* Blaisdell, New York, 1963. Reprint: Dover, 1975.

[5] P. Dewilde and H. Dym. Schur recursions, error formulas, and convergence of rational estimators for stationary stochastic sequences. *IEEE Trans. on Information Theory*, IT-27:446–461, 1981.

[6] M.M. Djrbashian. A survey on the theory of orthogonal systems and some open problems. In P. Nevai, editor, *Orthogonal polynomials: Theory and practice*, volume 294 of *Series C: Mathematical and Physical Sciences*, pages 135–146, Boston, 1990. NATO-ASI, Kluwer Academic Publishers.

[7] W.B. Jones, O. Njåstad, and W.J. Thron. Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle. *Bull. London Math. Soc.*, 21:113–152, 1989.

[8] R. Nevanlinna. Über beschränkte analytische Funktionen. *Ann. Acad. Sci. Fenn. Ser. A.*, 32(7):75pp., 1929.

[9] G. Pick. Über die Beschränkungen analytischen Funktionen Welche durch vorgegebene Funktionswerte bewirkt werden. *Math. Ann.*, 77:7–23, 1916.

[10] G. Pick. Über die Beschränkungen analytischen Funktionen durch vorgegebene Funktionswerte. *Math. Ann.*, 78:270–275, 1918.