A MOMENT PROBLEM ASSOCIATED TO RATIONAL SZEGŐ FUNCTIONS

A. Bultheel
Department of Computer Science, K.U. Leuven,
B-3001 Leuven, Belgium.

P. González-Vera
Facultad de Matemáticas, Universidad de La Laguna,
La Laguna, Tenerife, Canary Islands, Spain.

E. Hendriksen
University of Amsterdam, Department of Mathematics,
Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands.

O. Njåstad
Department of Mathematics, University of Trondheim-NTH,
N-7034 Trondheim, Norway.

ABSTRACT

Let $a_1, \ldots, a_p$ be distinct points in the finite complex plane $\mathbf{C}$, such that $|a_j| > 1$, $j = 1, \ldots, p$ and let $b_j = 1/\overline{a}_j$, $j = 1, \ldots, p$. Let μ_0, $\mu_n^{(j)}$, $\nu_n^{(j)}$, $j = 1, \ldots, p$; $n = 1, 2, \ldots$ be given complex numbers. We consider the following moment problem.

Find a distribution ψ on $[-\pi, \pi]$, with infinitely many points of increase, such that

$$\int_{-\pi}^{\pi} d\psi(\theta) = \mu_0,$$

$$\int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - a_j)^n} = \mu_n^{(j)}, \quad \int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - b_j)^n} = \nu_n^{(j)}, \quad j = 1, \ldots, p; \quad n = 1, 2, \ldots.$$

It will be shown that this problem has a unique solution if the moments generate a positive-definite Hermitian inner product on the linear space of rational functions with no poles in the extended complex plane $\mathbf{C}^*$ outside $\{a_1, \ldots, a_p, b_1, \ldots, b_p\}$.

KEYWORDS: moment problem, orthogonal rational functions, quasi-definite, positive-definite, Hermitian functional.

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1. Introduction. Recently certain rational functions orthogonal on the unit circle, arising in a natural way from the Pick-Nevanlinna interpolation problem, have been studied. The corresponding moment problem generalizes the classical trigonometric moment problem. In the present paper we consider the case where the moments are given in a finite number of different points.

Let the subsets D, E, ∂D of the (finite) complex plane $\mathbf{C}$ be given by

$$D = \{z : |z| < 1\}, \quad E = \{z : |z| > 1\}, \quad \partial D = \{z : |z| = 1\}.$$

Let $a_1, \ldots, a_p \in E$ and let $b_j = 1/\overline{a}_j$, $j = 1, \ldots, p$, where $a_i \neq a_j$ if $i \neq j$. Let $\{\mu_n^{(j)}\}_{n=0}^{\infty}$ and $\{\nu_n^{(j)}\}_{n=0}^{\infty}$ be given sequences of complex numbers with $\mu_0^{(j)} = \nu_0^{(j)} = \mu_0 = \nu_0 \neq 0$ for $j = 1, \ldots, p$.

MOMENT PROBLEM: Find a distribution ψ on the interval $[-\pi, \pi]$, with infinitely many points of increase, such that

$$\int_{-\pi}^{\pi} d\psi(\theta) = \mu_0,$$

$$\int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - a_j)^n} = \mu_n^{(j)}, \quad \int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - b_j)^n} = \nu_n^{(j)}, \quad j = 1, \ldots, p; \quad n = 1, 2, \ldots.$$

This moment problem will be referred to as the "Rational Szegő Moment Problem", in short RSMP.

Let $\mathcal{A}$ be the linear space of rational functions with no poles in the extended complex plane $\mathbf{C}^*$ outside the set $\{a_1, \ldots, a_p\}$ and let $\mathcal{B}$ be the linear space of rational functions with no poles in the extended complex plane $\mathbf{C}^*$ outside $\{b_1, \ldots, b_p\}$. Then the moments μ_0, $\mu_n^{(j)}$, $\nu_n^{(j)}$ give rise to a linear functional Φ on $\mathcal{R} := \mathcal{A} + \mathcal{B}$, defined by

$$\Phi(1) = \mu_0,$$

$$\Phi(\frac{1}{(z - a_j)^n}) = \mu_n^{(j)} \quad \text{and} \quad \Phi(\frac{1}{(z - b_j)^n}) = \nu_n^{(j)}, \quad j = 1, \ldots, p; \, n = 1, 2, \ldots. \tag{1.1}$$

Next we define a functional $< . , . >$ on $\mathcal{R} \times \mathcal{R}$ by

$$< X(z), Y(z) >= \Phi(X(z)\overline{Y}(z^{-1})). \tag{1.2}$$

Here $\overline{Y}(z^{-1}) = \overline{Y(1/\overline{z})}$. It is easily shown, see [4], that $< . , . >$ is *Hermitian*, i.e. $< X, Y >= \overline{< Y, X >}$ for all $X, Y \in \mathcal{R}$, if and only if

$$\mu_0 = \overline{\mu}_0 \quad \text{and} \quad \overline{\nu}_n^{(j)} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} a_j^{n+k} \mu_k^{(j)}, \quad j = 1, \ldots, p; n = 1, 2, \ldots.$$

In the present paper we say that the functional $< . , . >$ is *quasi-definite* on $\mathcal{A} \times \mathcal{A}$ if $< X, X > \neq 0$ for all $X \in \mathcal{A}$ with $X \neq 0$. If the functional $< . , . >$ is Hermitian, this functional is called *positive-definite* on $\mathcal{A} \times \mathcal{A}$ if $< X, X >> 0$ for all $X \in \mathcal{A}$ with $X \neq 0$.

If the RSMP has a solution, say ψ, then $< . , . >$ is a Hermitian functional on $\mathcal{R} \times \mathcal{R}$ which is positive-definite on $\mathcal{R} \times \mathcal{R}$. Indeed, if $X \in \mathcal{R}$, $X \neq 0$, then

$$\int_{-\pi}^{\pi} |X(e^{i\theta})|^2 \, d\psi(\theta) \geq 0$$

and the equality sign occurs only if X vanishes at all the points where ψ increases, which is impossible since X has only a finite number of zeros.

Conversely, if $< . , . >$, given by (1.2), while Φ generates the moments as in (1.1), is a Hermitian functional on $\mathcal{R} \times \mathcal{R}$ which is positive-definite on $\mathcal{A} \times \mathcal{A}$, then the existence of a solution to the RSMP is shown, using Helly's theorems, orthogonal rational Szegő functions and a Gaussian quadrature formula.

Let us assume that the functional $< . , . >$ is Hermitian on $\mathcal{R} \times \mathcal{R}$ and quasi-definite on $\mathcal{A} \times \mathcal{A}$. By partial fraction decomposition every function $A \in \mathcal{A}$ can be written as

$$A(z) = \alpha_0 + \sum_{j=1}^{p} \sum_{k=1}^{m_j} \frac{\alpha_{jk}}{(z - a_j)^k} \tag{1.3}$$

where $\alpha_0, \alpha_{jk} \in \mathbf{C}$ and $m_j \in \mathbf{N}$.

Functions $B \in \mathcal{B}$ can be written in a similar way with all the a_j replaced by b_j. Every natural number n can be written in a unique way as $n = q_n p + r_n$ where $1 \le r_n \le p$. We shall write q for q_n and r for r_n. When $r = p$ we mean a_1 by a_{r+1}, when $r = 1$ we mean a_p by a_{r-1} and so on. Statements related to a_{r+1}, a_{r-1}, etc. will be clear from the context. We define

$$s_j^{(n)} = s_j = \begin{cases} q+1 & \text{for} \quad j \le r, \\ q & \text{for} \quad j \ge r+1, \end{cases} \qquad j = 1, \ldots, p.$$

Notice that $s_1 + \ldots + s_p = n$. For fixed but arbitrary n the space $\mathcal{A}_n$ is defined to consist of all functions of the form (1.3) with $m_j = s_j$. The spaces $\mathcal{B}_n$ are defined analogously and $\mathcal{A}_n + \mathcal{B}_n$ is denoted by $\mathcal{R}_n$. We define polynomials $N_n(z;a)$ and $N_n(z;b)$ by

$$N_n(z;a) = \prod_{j=1}^{p} (z - a_j)^{s_j} \qquad \text{and} \qquad N_n(z;b) = \prod_{j=1}^{p} (z - b_j)^{s_j}.$$

Then $\mathcal{A}_n$ and $\mathcal{B}_n$ consist respectively of all functions of the form

$$A(z) = \frac{C(z)}{N_n(z;a)} \qquad \text{and} \qquad B(z) = \frac{D(z)}{N_n(z;b)},$$

where C and D are polynomials of degree at most n. If $\mathcal{S}$ is a subspace of $\mathcal{R}$, then the subspace of $\mathcal{S}$ consisting of all functions with constant term α_0 equal to zero, is denoted as $\mathcal{S}^0$.
Thus $A(z) = C(z)/N_n(z;a)$ belongs to $\mathcal{A}^0$ if and only if $\deg C(z) < n$. We observe that $A(z) \in \mathcal{A}_n$ implies $B(z) = \overline{A}(z^{-1}) \in \mathcal{B}_n$ and vice versa.
Since every $A \in \mathcal{A}_n \backslash \mathcal{A}_{n-1}$ has a unique representation

$$A(z) = \frac{P(z)}{N_n(z;a)}$$

where P is a polynomial with $\deg P \le n$, we may define the reciprocal function A^* by

$$A^*(z) = \frac{z^n \overline{P}(z^{-1})}{N_n(z;a)}.$$

Notice that $A^* \in \mathcal{A}_n$. If A and A^* are in $\mathcal{A}_n \backslash \mathcal{A}_{n-1}$, then $A^{**} = A$.

2. Rational Szegő functions. As we have assumed that the functional $< .\,,. >$ is Hermitian on $\mathcal{R} \times \mathcal{R}$ and quasi-definite on $\mathcal{A} \times \mathcal{A}$ we can apply the Gram-Schmidt process with respect to $< .\,,. >$ on the sequence

$$1, \frac{1}{z-a_1}, \ldots, \frac{1}{z-a_p}, \frac{1}{(z-a_1)^2}, \ldots, \frac{1}{(z-a_p)^2}, \frac{1}{(z-a_1)^3}, \ldots$$

to obtain an orthogonal sequence $\{\rho_n\}_{n=0}^{\infty}$ of functions in $\mathcal{A}$. These functions ρ_n are called *Rational Szegő functions*.

Clearly $\rho_n \in \mathcal{A}_n \backslash \mathcal{A}_{n-1}$ and $\rho_n \perp \mathcal{A}_{n-1}$. The ρ_n are of the form

$$\rho_n(z) = \frac{R_n(z)}{N_n(z;a)}$$

where R_n is a polynomial of degree at most n. The ρ_n can also be written as

$$\rho_n(z) = \beta_0^{(n)} + \frac{\beta_1^{(n)}}{z-a_1} + \ldots + \frac{\beta_p^{(n)}}{z-a_p} + \frac{\beta_{p+1}^{(n)}}{(z-a_1)^2} + \ldots + \frac{\beta_{n-1}^{(n)}}{(z-a_{r-1})^{q+1}} + \frac{\beta_n^{(n)}}{(z-a_r)^{q+1}},$$

where $\beta_n^{(n)} \ne 0$. In this paper we always assume that the ρ_n are *monic*, i.e. we assume that $\beta_n^{(n)} = 1$. We say that ρ_n is *regular* if $\beta_{n-1}^{(n)} \ne 0$. This means that a_{r-1} is not a zero of R_n. The *reciprocal rational Szegő functions* ρ_n^* are given by

$$\rho_n^*(z) = \frac{z^n \overline{R}_n(z^{-1})}{N_n(z;a)}, \quad n = 0, 1, \ldots.$$

For each n let

$$a^{(n)} = \prod_{j=1}^{p} a_j^{s_j} \quad \text{and} \quad b^{(n)} = \prod_{j=1}^{p} b_j^{s_j}.$$

Then the operation $S(z) \to \overline{S}(z^{-1})$ applied to $N_n(z;a)$ results in

$$\frac{(-1)^n}{b^{(n)}} z^{-n} N_n(z;b),$$

hence

$$\overline{N}_n(z^{-1};a) = \frac{(-1)^n}{b^{(n)}} z^{-n} N_n(z;b) \tag{2.1}$$

and similarly

$$\overline{N}_n(z^{-1};b) = \frac{(-1)^n}{a^{(n)}} z^{-n} N_n(z;a). \tag{2.2}$$

From [3] we quote

LEMMA 2.1. If $\rho \in \mathcal{A}_n$, $\rho(z) = R(z)/N_n(z;a)$ and $\tau(z) = z^n \overline{R}(z^{-1})/N_n(z;a)$, then

$$\rho \perp \mathcal{A}_{n-1} \iff \tau \perp (z - b_r)\mathcal{A}_n^0. \tag{2.3}$$

COROLLARY 2.1. $\rho_n^* \perp (z - b_r)\mathcal{A}_n^0$.

THEOREM 2.1. The rational Szegő functions ρ_n satisfy the recurrence relations

$$\rho_1 = \beta_1 \frac{z}{z - a_1} \rho_0 + \delta_1 \frac{1}{z - a_1} \rho_0^* \tag{2.4}$$

and

$$\rho_n = \beta_n \frac{z - b_{r-1}}{z - a_r} \rho_{n-1} + \delta_n \frac{z - a_{r-1}}{z - a_r} \rho_{n-1}^*, \quad n = 2, 3, \ldots. \tag{2.5}$$

If the ρ_n are regular, which is equivalent to $\beta_n \neq 0$ for all n, we also have, see [3,Th. 3.1]

THEOREM 2.2. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of functions such that each ρ_n is monic and belongs to $\mathcal{A}_n \setminus \mathcal{A}_{n-1}$ as $n \geq 1$ and $\rho_0 \in \mathcal{A}_0$. Assume that ρ_n and ρ_n^* satisfy (2.4) and (2.5) with $|\beta_1| > |\delta_1|$ and $|\beta_n| > |\delta_n a_{r-1}|$, $n = 2, 3, \ldots$. Then there exists a Hermitian functional $< .\,, . >$ on $\mathcal{R} \times \mathcal{R}$ with $< X(z), Y(z) > = < X(z)\overline{Y}(z^{-1}), 1 >$ for all $X, Y \in \mathcal{R}$, which is positive-definite on $\mathcal{A} \times \mathcal{A}$.

REMARK 2.1. Using (2.1) and (2.2) it is easily shown that the functional $< .\,, . >$ of Theorem 2.2 is also positive-definite on $\mathcal{R} \times \mathcal{R}$.

In the following we will use relations like

$$< \frac{z - b_i}{z - a_j} X(z), Y(z) > = \frac{b_i}{a_j} < X(z), \frac{z - a_i}{z - b_j} Y(z) >,$$

valid for all $X, Y \in \mathcal{R}$, without explicit reference.

3. Para-orthogonal rational functions. Recall that $< .\,, . >$ is Hermitian on $\mathcal{R} \times \mathcal{R}$ and quasi-definite on $\mathcal{A} \times \mathcal{A}$. Let

$$A_n(z) = \frac{1}{N_{n-1}(z;a)} \quad \text{and} \quad B_n(z) = \frac{z - b_r}{N_n(z;a)}.$$

A sequence $\{X_n\}_{n=0}^{\infty}$ of functions $X_n \in \mathcal{A}_n \setminus \mathcal{A}_{n-1}$, $n = 1, 2, \ldots$, $X_0 \in \mathcal{A}_0$, will be called *para-orthogonal* if

$$< X_n, A_n > \neq 0, \quad X_n \perp (z - b_r)\mathcal{A}_{n-1}^0, \quad < X_n, B_n > \neq 0,$$

for all relevant n. A function $X \in \mathcal{A}$ is called *κ-invariant* for some $\kappa \in \mathbf{C}$, $\kappa \neq 0$, if

$$X^*(z) = \kappa X(z).$$

Theorems 3.1 and 3.2 can be proved in a similar way as Theorems 2 and 3 in [2, section 3]. We omit the proofs here.

THEOREM 3.1. If $w_n \in \mathbf{C}$, $|w_n| = 1$, the the functions W_n defined by $W_n(z) = \rho_n(z) + w_n \rho_n^*(z)$, form a para-orthogonal sequence of $\overline{w}_n$-invariant rational functions.

REMARK 3.1. Clearly we have

$$\mathcal{A}_{n-1} = (z - b_r)\mathcal{A}_{n-1}^0 + \{\lambda A_n : \lambda \in \mathbf{C}\},$$

$$(z - b_r)\mathcal{A}_n^0 = (z - b_r)\mathcal{A}_{n-1}^0 + \{\lambda B_n : \lambda \in \mathbf{C}\},$$

$$\mathcal{A}_n = \{\lambda A_n : \lambda \in \mathbf{C}\} + (z - b_r)\mathcal{A}_{n-1}^0 + \{\lambda B_n : \lambda \in \mathbf{C}\}.$$

THEOREM 3.2. If $\{X_n\}_{n=0}^{\infty}$ is a para-orthogonal sequence of κ_n-invariant rational functions, then $|\kappa_n| = 1$ and the X_n are of the form

$$X_n(z) = \alpha_n(\rho_n(z) + w_n \rho_n^*(z))$$

with $\alpha_n \in \mathbf{C}$, $\alpha_n \neq 0$, and $w_n = \overline{\alpha}_n \overline{\kappa}_n / \alpha_n$.

Concerning the zeros of para-orthogonal rational functions we have

THEOREM 3.3. Let $\{X_n\}_{n=0}^{\infty}$ be a para-orthogonal sequence of κ_n-invariant rational functions. Then for each $n \geq 0$, the zeros of X_n are simple and lie on ∂D.
Proof. Let

$$X_n(z) = \frac{S_n(z)}{N_n(z;a)}, \quad S_n(z) = \sigma_0 + \sigma_1 z + \cdots + \sigma_n z^n.$$

Then

$$X_n^* = \frac{z^n \overline{S}_n(z^{-1})}{N_n(z;a)} = \kappa_n \frac{S_n(z)}{N_n(z;a)} = \kappa_n X_n(z),$$

so $\overline{\sigma}_{n-j} = \kappa_n \sigma_j$, $j = 0, 1, \ldots, n$. For $\ell \geq 0$ this implies that 0 is a zero of order ℓ of X_n if and only if ∞ is a zero of order ℓ of X_n.

Now let $\gamma \in \mathbf{C} \setminus \{0\}$. If γ is zero of order k of S_n, $(k \geq 0)$, then

$$X_n(z) = \frac{(z-\gamma)^k S_{n-k}(z)}{N_n(z;a)}$$

where S_{n-k} is a polynomial with degree at most $n-k$ and $S_{n-k}(\gamma) \neq 0$. As

$$\begin{aligned}\kappa_n \frac{(z-\gamma)^k S_{n-k}(z)}{N_n(z;a)} &= \kappa_n X_n(z) = X_n^*(z) = \frac{z^n \overline{S}_n(z^{-1})}{N_n(z;a)} \\ &= \frac{z^n (z^{-1} - \overline{\gamma})^k \overline{S}_{n-k}(z^{-1})}{N_n(z;a)} = (-1)^k \overline{\gamma}^k \frac{(z - \overline{\gamma}^{-1})^k z^{n-k} \overline{S}_{n-k}(z^{-1})}{N_n(z;a)}\end{aligned}$$

where

$$\left[\overline{S}_{n-k}(z^{-1})\right]_{z=\overline{\gamma}^{-1}} = \overline{S}_{n-k}(\overline{\gamma}) = \overline{S_{n-k}(\gamma)} \neq 0,$$

it follows that $\overline{\gamma}^{-1}$ is zero of order k of S_n.

Let $\alpha_1, \ldots, \alpha_s$ be the zeros of *odd order* of S_n which are on ∂D, $(s \geq 0)$. Suppose α_j has order $2m_j + 1$, $j = 1, \ldots, s$. Let X_n have zeros 0 and ∞ of order ℓ, and let the remaining zeros of S_n be $\beta_1, \beta_2, \ldots, \beta_t, \overline{\beta_1}^{-1}, \overline{\beta_2}^{-1}, \ldots, \overline{\beta_t}^{-1}$ repeated with multiplicity, $(t \geq 0)$. Set $m_1 + \cdots + m_s = M$. Then we may write

$$X_n(z) = \frac{z^\ell}{N_n(z;a)} \cdot \prod_{j=1}^{s} (z - \alpha_j)^{2m_j+1} \cdot \prod_{j=1}^{t} (z - \beta_j)(z - \overline{\beta_j}^{-1})$$

where $\ell + 2M + s + 2t = n - \ell$. Assume $\ell + M + t \geq 1$ and put

$$T(z) = \frac{z^{\ell+M+t-1}(z - b_r)}{N_{n-1}(z;a)} \cdot \prod_{j=1}^{s} (z - \alpha_j).$$

Since $\ell + M + t - 1 + 1 + s = \ell + M + s + t = n - \ell - M - t \leq n - 1$ we have $T \in (z - b_r)\mathcal{A}^0_{n-1}$, hence $< X_n, T >= 0$.

On the other hand

$$
\begin{aligned}
< X_n, T > &= < \frac{z^\ell \prod_{j=1}^s (z-\alpha_j)^{2m_j+1} \cdot \prod_{j=1}^t (z-\beta_j)(z-\overline{\beta}_j^{-1})}{N_n(z;a)}, \\
&\qquad \frac{z^{\ell+M+t-1}(z-b_r)\prod_{j=1}^s (z-\alpha_j)}{N_{n-1}(z;a)} > \\
&= < \frac{z(z^{-1}-\overline{b}_r)\prod_{j=1}^s (z-\alpha_j)^{m_j+1} \cdot \prod_{j=1}^t (z-\beta_j)}{N_n(z;a)}, \\
&\qquad \frac{z^{M+t}\prod_{j=1}^s (z^{-1}-\overline{\alpha}_j)^{m_j}(z-\alpha_j) \cdot \prod_{j=1}^t (z^{-1}-\beta_j^{-1})}{N_{n-1}(z;a)} > \\
= \frac{(-1)^{M+t+1}\overline{b}_r \prod_{j=1}^s \alpha_j^{m_j}}{\prod_{j=1}^t \overline{\beta}_j} &< \frac{\prod_{j=1}^s (z-\alpha_j)^{m_j+1} \cdot \prod_{j=1}^t (z-\beta_j)}{N_{n-1}(z;a)}, \\
&\qquad \frac{\prod_{j=1}^s (z-\alpha_j)^{m_j+1} \cdot \prod_{j=1}^t (z-\beta_j)}{N_{n-1}(z;a)} > \neq 0,
\end{aligned}
$$

since $< .\,, . >$ is quasi-definite on $\mathcal{A} \times \mathcal{A}$. Contradiction. Hence $\ell + M + t = 0$. This means that all the zeros of X_n are simple and lie on ∂D. □

4. Quadrature. Let $w_n \in \mathbf{C}$, $|w_n| = 1$, $n = 1, 2, \ldots$. We consider the para-orthogonal sequence $\{W_n(z; w_n)\}_{n=0}^\infty$ of w_n-invariant rational functions

$$W_n(z; w_n) = \rho_n(z) + w_n \rho_n^*(z), \quad n = 0, 1, 2, \ldots. \tag{4.1}$$

By Theorem 3.3 the zeros $\alpha_j = \alpha_j^{(n)}$, $j = 1, \ldots, n$, of W_n are simple and lie on ∂D. Hence there are nonzero $c_n \in \mathbf{C}$ such that

$$W_n(z; w_n) = \frac{c_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{N_n(z;a)}, \quad n = 0, 1, 2, \ldots.$$

For $i = 1, \ldots, n$ and $n = 1, 2, \ldots$ let

$$P_{n,i}(z) = \frac{(z-\alpha_1)\cdots(z-\alpha_{i-1})(z-\alpha_{i+1})\cdots(z-\alpha_n)}{(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)}.$$

Then

$$P_{n,i}(z) = \frac{W_n(z; w_n)}{(z-\alpha_i)W_n'(\alpha_i, w_n)}$$

and obviously

$$P_{n,i}(\alpha_j) = \delta_{i,j}, \quad i, j = 1, \ldots, n; n = 1, 2, \ldots.$$

Next we put

$$\Lambda_{n,i}(z) = N_{n-1}(\alpha_i; a)\frac{P_{n,i}(z)}{N_{n-1}(z;a)}, \quad i = 1, \ldots, n; n = 1, 2, \ldots.$$

Then $\Lambda_{n,i} \in \mathcal{A}_{n-1}$ and again $\Lambda_{n,i}(\alpha_j) = \delta_{i,j}$, $i, j = 1, \ldots, n; n = 1, 2, \ldots$.

Now take $R \in \mathcal{R}_{n-1}$ arbitrarily. Then

$$R(z) = \frac{\pi_{2n-2}(z)}{N_{n-1}(z;a)N_{n-1}(z;b)}$$

where π_{2n-2} is a polynomial with $deg\pi_{2n-2} \leq 2n-2$, and the function

$$F(z) = R(z) - \sum_{i=1}^n R(\alpha_i)\Lambda_{n,i}(z)$$

belonging to $\mathcal{R}_{n-1}$, may be written as

$$F(z) = \frac{Q_{2n-2}}{N_{n-1}(z;a)N_{n-1}(z;b)},$$

where

$$Q_{2n-2}(z) = \pi_{2n-2}(z) - \sum_{i=1}^{n} R(\alpha_i)N_{n-1}(\alpha_i;a)N_{n-1}(z;b)P_{n,i}(z).$$

Clearly Q_{2n-2} is a polynomial with $deg Q_{2n-2} \leq 2n-2$. As $Q_{2n-2}(\alpha_j) = 0$, $j = 1,\ldots,n$, there is a polynomial π_{n-2} of degree at most $n-2$ such that

$$\begin{aligned} F(z) &= \frac{c_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)}{N_n(z;a)} \cdot \frac{(z-a_r)\pi_{n-2}(z)}{N_{n-1}(z;b)} \\ &= W_n(z;w_n) \cdot \frac{(z-a_r)\pi_{n-2}(z)}{N_{n-1}(z;b)}. \end{aligned}$$

If

$$U(z) = \frac{(z-a_r)\pi_{n-2}(z)}{N_{n-1}(z;b)} \quad \text{and} \quad S(z) = \overline{U}(z^{-1})$$

then $S \in (z-b_r)\mathcal{A}_{n-1}^0$, so by para-orthogonality,

$$< F(z), 1 >=< W_n(z;w_n)U(z), 1 >=< W_n(z;w_n), S(z) >= 0.$$

With

$$(4.2) \qquad < \Lambda_{n,i}(z), 1 >= \lambda_{n,i}, \quad i = 1,\ldots,n; n = 1,2,\ldots$$

this implies

$$(4.3) \qquad < R(z), 1 >= \sum_{i=1}^{n} R(\alpha_i)\,\lambda_{n,i}, \quad R \in \mathcal{R}_{n-1},$$

a quadrature formula for $\mathcal{R}_{n-1}$. As $\Lambda_{n,i}(z)\overline{\Lambda}_{n,i}(z^{-1}) - \Lambda_{n,i}(z) \in \mathcal{R}_{n-1}$ it follows from (4.3) and $\alpha_i \in \partial D$ that

$$\begin{aligned} < \Lambda_{n,i}(z)\overline{\Lambda}_{n,i}(z^{-1}) - \Lambda_{n,i}(z), 1 > &= \sum_{j=1}^{n}[\Lambda_{n,i}(\alpha_j)\overline{\Lambda}_{n,i}(\overline{\alpha}_j) - \Lambda_{n,i}(\alpha_j)]\lambda_{n,j} \\ &= [|\Lambda_{n,i}(\alpha_i)|^2 - \Lambda_{n,i}(\alpha_i)]\lambda_{n,i} = 0. \end{aligned}$$

Hence

$$\lambda_{n,i} =< \Lambda_{n,i}(z), 1 >=< \Lambda_{n,i}(z)\overline{\Lambda}_{n,i}(z^{-1}), 1 >=< \Lambda_{n,i}(z), \Lambda_{n,i}(z) >\neq 0$$

since $< .,. >$ is quasi-definite on $\mathcal{A} \times \mathcal{A}$. If $< .,. >$ is positive-definite on $\mathcal{A} \times \mathcal{A}$, then $\lambda_{n,i} > 0$. We summarize the results of this section in

THEOREM 4.1. Let the functional $< .,. >$ be Hermitian on $\mathcal{R} \times \mathcal{R}$ and quasi-definite on $\mathcal{A} \times \mathcal{A}$. Let $\alpha_1^{(n)},\ldots,\alpha_n^{(n)}$ be the zeros of $W_n(z;w_n)$ defined by (4.1) and let the complex numbers $\lambda_{n,1},\ldots,\lambda_{n,n}$ be given by (4.2). Then $\lambda_{n,i} \neq 0$, $i = 1,\ldots,n$ and

$$< R(z), 1 >= \sum_{i=1}^{n} R(\alpha_i^{(n)})\,\lambda_{n,i}$$

for all $R \in \mathcal{R}_{n-1}$. If in addition $< .,. >$ is positive-definite on $\mathcal{A} \times \mathcal{A}$, then $\lambda_{n,i} > 0$, $i = 1,\ldots,n$.

REMARK 4.1. It follows also from Theorem 4.1 that a functional $< .,. >$ which is Hermitian on $\mathcal{R} \times \mathcal{R}$ and positive-definite on $\mathcal{A} \times \mathcal{A}$, is also positive-definite on $\mathcal{R} \times \mathcal{R}$.

5. The moment problem. In the remaining part of the paper we assume that the functional $< .,. >$ is Hermitian on $\mathcal{R} \times \mathcal{R}$ and *positive-definite* on $\mathcal{A} \times \mathcal{A}$. Again let $\alpha_1, \ldots, \alpha_n$ be the zeros of the function $W_n(z; w_n)$ with $|w_n| = 1$. Let $\alpha_j = e^{i\theta_j}$, $\theta \in [-\pi, \pi)$, $j = 1, \ldots, n$. Let these zeros be ordered such that

$$-\pi \leq \theta_1 < \theta_2 < \ldots < \theta_n < \pi.$$

Let $\lambda_{n,i}$, $i = 1, \ldots, n$, be given by (4.2). For each n we define distributions ψ_n on $[-\pi, \pi)$ by

$$\psi_n(\theta) = \begin{cases} 0 & \text{if} \quad -\pi \leq \theta \leq \theta_1, \\ \sum_{j=1}^{k} \lambda_{n,j} & \text{if} \quad \theta_k < \theta \leq \theta_{k+1}, \quad k = 1, 2, \ldots, n-1, \\ \mu_0 & \text{if} \quad \theta_n < \theta \leq \pi. \end{cases}$$

Then clearly

$$\Phi(R) = < R(z), 1 > = \int_{-\pi}^{\pi} R(e^{i\theta}) d\psi_n(\theta) \quad \text{if} \quad R \in \mathcal{R}_{n-1}.$$

By Helly's selection principle there is an increasing sequence $\{n_s\}_{s=1}^{\infty}$ and a non-decreasing function ψ on $[-\pi, \pi]$ such that

$$\lim_{s \to \infty} \psi_{n_s}(\theta) = \psi(\theta) \quad \text{for all} \quad \theta \in [-\pi, \pi].$$

By Helly's theorem we have

$$\int_{-\pi}^{\pi} f(\theta) d\psi_{n_s}(\theta) \to \int_{-\pi}^{\pi} f(\theta) d\psi(\theta) \quad \text{for continuous} \quad f \quad \text{on} \quad [-\pi, \pi].$$

Now

$$\frac{1}{(z - a_j)^k} \in \mathcal{A}_{n_s - 1} \quad \text{and} \quad \frac{1}{(z - b_j)^k} \in \mathcal{B}_{n_s - 1} \quad \text{if} \quad n_s \geq (k-1)p + j + 1,$$

so

$$\int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - a_j)^k} = \lim_{s \to \infty} \int_{-\pi}^{\pi} \frac{d\psi_{n_s}(\theta)}{(e^{i\theta} - a_j)^k} = \mu_k^{(j)}$$

and

$$\int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - b_j)^k} = \lim_{s \to \infty} \int_{-\pi}^{\pi} \frac{d\psi_{n_s}(\theta)}{(e^{i\theta} - b_j)^k} = \nu_k^{(j)}$$

as $j = 1, \ldots, p$; $k = 1, 2, \ldots$. Obviously also

$$\int_{-\pi}^{\pi} d\psi(\theta) = \lim_{s \to \infty} \int_{-\pi}^{\pi} d\psi_{n_s}(\theta) = \mu_0.$$

Next we show that ψ has an infinite number of points of increase. Suppose that ψ has only a finite number of such points, say $t_1, \ldots, t_m$. Let

$$R(z) = (z - e^{it_1})(z - e^{it_2}) \cdots (z - e^{it_m})$$

and let

$$X(z) = \frac{R(z)}{N_m(z; a)}.$$

Then $0 \neq X \in \mathcal{A}$ and R and N_m have no common zeros. Hence

$$< X, X > = \int_{-\pi}^{\pi} \frac{R(e^{i\theta})}{N_m(e^{i\theta}; a)} \cdot \frac{\overline{R}(e^{-i\theta})}{\overline{N}_m(e^{-i\theta}; a)} d\psi(\theta) = \int_{-\pi}^{\pi} |\frac{R(e^{i\theta})}{N_m(e^{i\theta}; a)}|^2 d\psi(\theta) = 0.$$

Contradiction. Thus we have proved

THEOREM 5.1. The Rational Szegő Moment Problem has a solution if and only if the functional $< .,. >$ is Hermitian on $\mathcal{R} \times \mathcal{R}$ and positive-definite on $\mathcal{A} \times \mathcal{A}$.

Finally we show that the RSMP has (essentially) a unique solution. Two solutions are considered to be equal if their difference is a constant at all points where they are continuous. Let ψ be a solution to the RSMP. Let $\Gamma = \{e^{i\theta} : -\pi \le \theta < \pi\}$ and let $g : \Gamma \to \mathbf{R}$ be defined by $g(e^{i\theta}) = \psi(\theta)$. Since g is of bounded variation, the function

$$G(z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{\zeta - z}d\zeta, \quad |z| \neq 1$$

is analytic in D and in E. By integration by parts we obtain

$$\begin{aligned}
\mu_n^{(j)} &= \int_{-\pi}^{\pi} \frac{d\psi(\theta)}{(e^{i\theta} - a_j)^n} = \frac{\psi(\theta)}{(e^{i\theta} - a_j)^n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-nie^{i\theta}\psi(\theta)d\theta}{(e^{i\theta} - a_j)^{n+1}} \\
&= (-1)^n \frac{\psi(\pi) - \psi(-\pi)}{(1 + a_j)^n} + n\int_{-\pi}^{\pi} \frac{ie^{i\theta}\psi(\theta)d\theta}{(e^{i\theta} - a_j)^{n+1}} \\
&= \frac{(-1)^n}{(1 + a_j)^n}\mu_0 + n\int_\Gamma \frac{g(\zeta)d\zeta}{(\zeta - a_j)^{n+1}} \\
&= \frac{(-1)^n}{(1 + a_j)^n}\mu_0 + \frac{2\pi i}{(n-1)!}G^{(n)}(a_j),
\end{aligned}$$

where $G^{(n)}$ denotes the n-th derivative of G. Hence

$$\mu_n^{(j)} - \frac{(-1)^n}{(1 + a_j)^n}\mu_0 = \frac{2\pi i}{(n-1)!}G^{(n)}(a_j), \quad j = 1, \dots, p; n = 1, 2, \dots. \tag{5.1}$$

In the same way we get

$$\nu_n^{(j)} - \frac{(-1)^n}{(1 + b_j)^n}\mu_0 = \frac{2\pi i}{(n-1)!}G^{(n)}(b_j), \quad j = 1, \dots, p; n = 1, 2, \dots. \tag{5.2}$$

Now let ϕ be another solution to the RSMP. Let $f(e^{i\theta}) = \phi(\theta)$, $-\pi \le \theta < \pi$ and set

$$F(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)d\zeta}{\zeta - z}, \quad |z| \neq 1.$$

Then F is analytic in D and in E and also

$$\mu_n^{(j)} - \frac{(-1)^n}{(1 + a_j)^n}\mu_0 = \frac{2\pi i}{(n-1)!}F^{(n)}(a_j), \quad j = 1, \dots, p; n = 1, 2, \dots. \tag{5.3}$$

and

$$\nu_n^{(j)} - \frac{(-1)^n}{(1 + b_j)^n}\mu_0 = \frac{2\pi i}{(n-1)!}F^{(n)}(b_j), \quad j = 1, \dots, p; n = 1, 2, \dots. \tag{5.4}$$

For the differences $H(z) = G(z) - F(z)$, $h(\zeta) = g(\zeta) - f(\zeta)$ and $\eta(\theta) = \psi(\theta) - \phi(\theta)$ we have

$$H(z) = \frac{1}{2\pi i}\int_\Gamma \frac{h(\zeta)d\zeta}{\zeta - z}$$

is analytic in D and in E, and by (5.1)-(5.4),

$$H^{(n)}(a_j) = H^{(n)}(b_j) = 0, \quad j = 1, \dots, p; n = 1, 2, \dots.$$

This implies that H is constant on D and on E. So for $z \in E$ we have

$$\begin{aligned}
H(z) &= \frac{1}{2\pi i}\int_\Gamma \frac{h(\zeta)d\zeta}{\zeta - z} = \frac{-1}{2\pi iz}\int_\Gamma \frac{h(\zeta)d\zeta}{1 - z^{-1}\zeta} \\
&= \frac{-1}{2\pi iz}\int_\Gamma h(\zeta)\sum_{k=0}^{\infty}\zeta^k z^{-k}d\zeta = -\sum_{k=0}^{\infty} z^{-k-1}\frac{1}{2\pi i}\int_\Gamma \zeta^k h(\zeta)d\zeta \\
&= -\sum_{k=0}^{\infty} z^{-k-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta(k+1)}\eta(\theta)d\theta.
\end{aligned}$$

Since H is constant on E it follows that the Fourier coefficients

$$c_{-k-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(k+1)} \eta(\theta) d\theta$$

of η are zero for $k = 0, 1, 2, \ldots$. For $z \in D$ we get in a similar way

$$\begin{aligned} H(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{h(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i z} \int_{\Gamma} \frac{\zeta^{-1} h(\zeta) d\zeta}{1 - z\zeta^{-1}} \\ &= -\sum_{k=0}^{\infty} z^k \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta k} \eta(\theta) d\theta, \end{aligned}$$

and since H is constant on D also the Fourier coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta k} \eta(\theta) d\theta$$

of η are zero for $k = 1, 2, \ldots$. Hence η is a constant function almost everywhere, with respect to the Lebesgue measure. This implies that ψ and ϕ are equal solutions. Combining the above result and Theorem 5.1 we obtain

THEOREM 5.2. The Rational Szegő Moment Problem has a unique solution if and only if the functional $< .\,, . >$ is Hermitian on $\mathcal{R} \times \mathcal{R}$ and positive-definite on $\mathcal{A} \times \mathcal{A}$.

REMARK 5.1. It is well known that the trigonometric moment problem always has a unique solution if the given sequence of moments is positive, (see [1]).

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