A general module theoretic framework for vector M-Pad and matrix rational interpolation

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Abstract

A general module theoretic framework is used to solve several classical interpolation problems and generalizations thereof in a unified way. These problems are divided into two main families. The first family contains the classical linearized Pad, Pad-Hermite and M-Pad problems and the generalization to the vector M-Pad problem. The second family consists of the Pad problem, the scalar, vector and matrix rational interpolation problems. The solution method is straightforward, recursive and efficient. It can follow any “path” in the “solution table” even if this “solution table” is nonnormal (nonperfect). Reordering of the interpolation data is not required.

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A general module theoretic framework for vector M-Pad and matrix rational interpolation

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A general module theoretic framework is used to solve several classical interpolation problems and generalizations thereof in a unified way. These problems are divided into two main families. The first family contains the classical linearized Pad, Pad-Hermite and M-Pad problems and the generalization to the vector M-Pad problem. The second family consists of the Pad problem, the scalar, vector and matrix rational interpolation problems. The solution method is straightforward, recursive and efficient. It can follow any “path” in the “solution table” even if this “solution table” is nonnormal (nonperfect). Reordering of the interpolation data is not required.

1 Introduction

In our earlier papers [7, 8, 2, 9], we have given recursive and efficient algorithms to solve several classical interpolation problems. In this paper, we show that all these classical problems and generalizations thereof can be solved in a unified way within the framework of module theory. Module theory has a long tradition in system theory. We refer to the last chapter of the book of Kalman, Falb and Arbib [5] where they use module theory to solve the minimal partial realization problem for matrix sequences. This problem is related to a matrix rational interpolation problem around infinity. In [3], Dickinson, Morf and Kailath give a recursive algorithm to solve this problem without explicitly referring to module theory. However, their approach already contains several ingredients also used in this paper. In recent years, module theoretic results are also appearing in the rational approximation field. For example, for the special case of M-Pad approximation, Beckermann [1, Thms. 3.1–3.4 and Eqn. (12)] derives the same results as in Theorems 3.1, 3.2 and 4.1 of this paper. Because the length of this paper is limited, we only include hints for the proofs and refer the interested reader to the above-mentioned papers for an introduction and further references to the classical interpolation problems. We shall consider submodules of the free module $\mathbf{F}[z]^m$ of the polynomial m -tuples over the principal ideal domain $\mathbf{F}[z]$, the set of the polynomials over the field $\mathbf{F}$. We shall concentrate our attention on two main families of interpolation problems:

1. The first family consists of the vector M-Pad approximation problems which are generalizations of the scalar problem, the Pad-Hermite and linearized Pad approximation problem.
2. The second family consists of matrix rational interpolation problems with as special cases the vector, scalar interpolation and Pad problems.

2 Basic formulations of the interpolation problems considered

Let us give the basic formulations of each of these two families.

1. For the first family, we consider the following vector M-Pad approximation problem. We are looking for all polynomial m -tuples $(p_1, p_2, \dots, p_m)$ such that
 - (a) they satisfy some vector interpolation conditions for $z \rightarrow \alpha$

$$f_1^{(\alpha)} p_1 + f_2^{(\alpha)} p_2 + \dots + f_m^{(\alpha)} p_m = \begin{bmatrix} O((z - \alpha)^{\sigma_1^{(\alpha)}}) \\ O((z - \alpha)^{\sigma_2^{(\alpha)}}) \\ \vdots \\ O((z - \alpha)^{\sigma_p^{(\alpha)}}) \end{bmatrix}, \forall \alpha \in I \quad (2.1)$$

with

- I a finite set of interpolation points $\alpha \in \mathbf{F}$,
- $f_j^{(\alpha)}$ a formal p -vector Taylor series around α ,
- $\sigma_i^{(\alpha)} \in \mathbf{N} = \{0, 1, 2, \dots\}$ the order indices for the interpolation point α which can be different for each row index i ;

(b) they satisfy some constraints on their degree-structure:

$$\deg p_i \leq \alpha_i, \quad i = 1, 2, \dots, m; \quad (2.2)$$

(c) the number of parameters (coefficients) is one more than the number of interpolation conditions:

$$\sum_{\alpha \in I} \sum_{i=1}^p \sigma_i^{(\alpha)} = \sum_{i=1}^m (\alpha_i + 1) - 1;$$

hence, there always exists a solution (in the normal case there exists a unique solution up to a scalar multiple).

2. For the second family, we consider the following matrix rational interpolation problem. Find all $(p \times q)$ rational functions Z such that

(a) they satisfy some matrix rational interpolation conditions

$$Z = T^{(\alpha)} + O((z - \alpha)^{\sigma_i^{(\alpha)}}), \quad \forall \alpha \in I, \quad (2.3)$$

with $T^{(\alpha)}$ a $(p \times q)$ Taylor series around α and the order indices $\sigma_i^{(\alpha)}$ dependent on the row index i ;

(b) they have smallest $\vec{s}$ -McMillan degree. The $\vec{s}$ -McMillan degree with $\vec{s} = (s_1, s_2, \dots, s_{p+q})$ and $s_i \in \mathbf{Z}$ (the set of integer numbers) of a rational function Z is defined as follows:

- take a right coprime polynomial matrix fraction description of Z : ND^{-1} (exists always but is not unique [4, p. 441, Th. 6.5–4]);
- take all possible $(q \times q)$ determinants from

$$\begin{bmatrix} z^{s_1} & & \\ & \ddots & \\ & & z^{s_{p+q}} \end{bmatrix} \begin{bmatrix} N \\ D \end{bmatrix};$$

- the greatest degree of all these determinants is called the $\vec{s}$ -McMillan degree.

Note that we get the classical McMillan degree with $\vec{s} = \vec{0}$. The McMillan degree is a well-known concept in linear system theory. It is a measure of the complexity of a linear system (see e.g. [4, p. 439]). The set of all rational functions Z satisfying the interpolation conditions (2.3) is not empty. From this set, we choose the optimal elements, i.e. the ones with smallest $\vec{s}$ -McMillan degree. Hence, there always exists a solution (not necessarily unique).

Note that more complicated formulations for these two families are possible, when adding additional constraints (e.g. pole conditions, stability conditions, ...).

Let us transform the last problem into a problem involving polynomial m -tuples with $m = p + q$. Rewrite Z as $Z = ND^{-1}$ with $D \in \mathbf{F}[z]^{q \times q}$. Multiplying (2.3) by D implies that a column n of the numerator polynomial matrix N and the corresponding column d of the denominator polynomial matrix D satisfy

$$\begin{bmatrix} I_{p \times p} & -T^{(\alpha)} \end{bmatrix} \begin{bmatrix} n \\ d \end{bmatrix} = \begin{bmatrix} O((z - \alpha)^{\sigma_1^{(\alpha)}}) \\ O((z - \alpha)^{\sigma_2^{(\alpha)}}) \\ \vdots \\ O((z - \alpha)^{\sigma_p^{(\alpha)}}) \end{bmatrix}, \quad \forall \alpha \in I. \quad (2.4)$$

If we interpret (n^T, d^T) as a polynomial m -tuple $(p_1, p_2, \dots, p_m)$ and the columns of the matrix Taylor series

$$\begin{bmatrix} I_{p \times p} & -T^{(\alpha)} \end{bmatrix}$$

as m vector Taylor series $f_1^{(\alpha)}, f_2^{(\alpha)}, \dots, f_m^{(\alpha)}$, we see that the interpolation conditions (2.1) and (2.4) are similar. To be able to go from (2.4) back to (2.3), we show by contradiction that $D(\alpha)$ is regular for $\alpha \in I$. Suppose that $D(\alpha)$ is singular. Hence, there exists a regular matrix $T \in \mathbf{F}^{q \times q}$ such that the last column d' of $D(\alpha)T$ is the

zero column. Note that $ND^{-1} = N'D'^{-1} = (NT)(DT)^{-1}$ represent the same rational function Z . Multiplying (2.3) by $D' = DT$ and taking the last columns of N' and D' , we get

$$\begin{bmatrix} I_{p \times p} & -T^{(\alpha)} \\ 0_{q \times p} & I_{q \times q} \end{bmatrix} \begin{bmatrix} n' \\ d' \end{bmatrix} = \begin{bmatrix} O((z - \alpha)^{\sigma_i^{(\alpha)} + 1}) \\ 0_{q \times 1} \end{bmatrix}$$

or, evaluated for $z = \alpha$,

$$\begin{bmatrix} I_{p \times p} & -T^{(\alpha)}(\alpha) \\ 0_{q \times p} & I_{q \times q} \end{bmatrix} \begin{bmatrix} n'(\alpha) \\ d'(\alpha) \end{bmatrix} = \begin{bmatrix} 0_{p \times 1} \\ 0_{q \times 1} \end{bmatrix}.$$

Hence, the polynomial matrix

$$\begin{bmatrix} N'(\alpha) \\ D'(\alpha) \end{bmatrix} = \begin{bmatrix} N(\alpha) \\ D(\alpha) \end{bmatrix} T$$

has not full rank. Therefore, $\text{diag}\{1, 1, \dots, (z - \alpha)\}$ is a non-trivial right common divisor of N and D . However, we assumed D and N to be right coprime. Hence, our assumption is wrong and $D(\alpha)$, $\forall \alpha \in I$, has to be of full rank. The $D(\alpha)$ being regular allows us to give the following equivalent reformulation of the original matrix rational interpolation problem of the second family. We have to look for q polynomial m -tuples (n_j^T, d_j^T) satisfying equation (2.4) with the additional conditions:

- the matrices $D(\alpha) = [d_1(\alpha) \cdots d_q(\alpha)]$, are regular, $\forall \alpha \in I$;
- the $\vec{s}$ -McMillan degree of $Z = ND^{-1}$ is as small as possible.

3 General module theoretic framework

Because the set of all polynomial m -tuples is a finitely generated free module over the principal ideal domain of the polynomials $\mathbf{F}[z]$, we know that each submodule is also free with dimension smaller than or equal to the dimension m of the complete module [6, pp. 358]. The key idea in this paper is the following fact.

Theorem 3.1 *The set S of all polynomial m -tuples $\vec{p} = (p_1, p_2, \dots, p_m)$ satisfying equations (2.1) forms a submodule of the $\mathbf{F}[z]$ -module $\mathbf{F}[z]^m$. A basis for the submodule S always consists of exactly m elements, i.e. the dimension of S is exactly equal to m .*

Proof. It is easy to see that $\vec{p}_1 + \vec{p}_2 \in S$, $\forall \vec{p}_1, \vec{p}_2 \in S$ and $c\vec{p} \in S$, $\forall \vec{p} \in S$, $\forall c \in \mathbf{F}[z]$. Hence, we have proven that S is a submodule. To prove the second part of the theorem, we refer to the recursive method of Section 4 keeping always just m basis vectors. $\square$

Within this submodule, we have to look for those polynomial m -tuples which satisfy additional conditions concerning their degree-structure. Therefore we need the concepts of $\vec{s}$ -degree and $\vec{s}$ -reducedness. The $\vec{s}$ -degree of a polynomial m -tuple $(p_1, p_2, \dots, p_m)$ is defined as a generalization of the classical degree:

$$\vec{s}\text{-deg}(p_1, p_2, \dots, p_m) = \max\{\deg p_i + s_i\}$$

with $\deg 0 = -\infty$. Different shift parameters s_i give different importance to the degree of the components of a polynomial m -tuple. The $\vec{s}$ -highest degree coefficient of a polynomial m -tuple $(p_1, p_2, \dots, p_m)$ with $\vec{s}$ -degree δ is the m -tuple whose i -th component is the coefficient of $z^{\delta - s_i}$ in p_i . The $\vec{s}$ -highest degree coefficient of the zero m -tuple is defined as the zero m -vector. A set of polynomial m -tuples is $\vec{s}$ -reduced if the $\vec{s}$ -highest degree coefficients are linearly independent. It is easy to transform the basis of a submodule into an $\vec{s}$ -reduced one. If the basis is not already $\vec{s}$ -reduced, take the basis element $\vec{b}_i$ with largest $\vec{s}$ -degree δ_i whose $\vec{s}$ -highest degree coefficient is linearly dependent on the $\vec{s}$ -highest degree coefficients (hdc) of the other basis elements

$$\sum_{j=1}^m d_j(\vec{s}\text{-hdc } \vec{b}_j) = 0$$

with $d_i \neq 0$. Replacing $\vec{b}_i$ by $\sum_{j=1}^m d_j z^{\delta_i - \delta_j} \vec{b}_j$, gives us another basis for S but the $\vec{s}$ -degree of $\vec{b}_i$ is decreased. Repeating this process, we finally get an $\vec{s}$ -reduced basis. The transformation, done in each step, can be represented as a unimodular matrix, i.e. a polynomial matrix with a nonzero determinant $\in \mathbf{F}$. This transformation matrix is left multiplied to the matrix whose rows are the basis polynomial m -tuples. The product of all these unimodular matrices is of course again unimodular. For further details, we refer to [10]. Once we have a basis in $\vec{s}$ -reduced form, it is easy to parametrize all elements of the submodule with an upper bound on the $\vec{s}$ -degree.

Theorem 3.2 *Given an $\vec{s}$ -reduced basis for a submodule S with basis elements $\vec{b}_i$ having $\vec{s}$ -degree δ_i , all elements of the submodule S having $\vec{s}$ -degree $\leq \delta$ can be parametrized uniquely as $\sum_{i=1}^m c_i \vec{b}_i$ with c_i a polynomial of degree $\leq \delta - \delta_i$.*

Proof. The $\vec{s}$ -highest degree coefficient of $\sum_{i=1}^m c_i \vec{b}_i$ is determined by

$$\sum_{\gamma_j + \delta_j = \max\{\gamma_i + \delta_i\}} (\text{hdc } c_j)(\vec{s}\text{-hdc } \vec{b}_j)$$

with $\gamma_i = \deg c_i$. This coefficient can not become zero because the $\vec{s}$ -highest degree coefficients of the basis elements are linearly independent. Hence,

$$\vec{s}\text{-deg}\left(\sum_{i=1}^m c_i \vec{b}_i\right) = \max\{\gamma_i + \delta_i\}.$$

This proves the theorem. $\square$

Note that Theorem 3.2 for Padé-Hermite approximants is similar to [9, Th. 3.3] and for scalar M-Padé approximation coincides with [1, Corollary 3.4].

From the previous section, we know that all solutions Z of the matrix rational interpolation problem can be written as $Z = ND^{-1}$ with N and D right coprime, and $D(\alpha)$ having full rank, $\forall \alpha \in I$. To handle this last condition, we need the following theorem.

Theorem 3.3 *Let $[n_i^T \ d_i^T]$, $i = 1, \dots, m$ be a basis for the submodule of all polynomial m -tuples that solve (2.1). Assume $p < m$ and $[f_1^{(\alpha)}(\alpha) \cdots f_p^{(\alpha)}(\alpha)]$ is regular, $\forall \alpha \in I$. Then the $q \times m$ matrix $[d_1(\alpha) \cdots d_m(\alpha)]$ has full rank $\forall \alpha \in I$.*

Proof. $\forall \alpha \in I$, let us define $\sigma_{\max}^{(\alpha)} = \max\{\sigma_i^{(\alpha)} : i = 1, 2, \dots, p\}$. $N_{(\alpha)}$ is defined as the $(p \times q)$ -matrix polynomial of degree $\leq \sigma_{\max}^{(\alpha)} - 1$ such that

$$N_{(\alpha)} = - \left[f_1^{(\alpha)} \cdots f_p^{(\alpha)} \right]^{-1} \left[f_{p+1}^{(\alpha)} \cdots f_m^{(\alpha)} \right] + O((z - \alpha)^{\sigma_{\max}^{(\alpha)}}).$$

It is clear that the columns of $\prod_{\beta \in I \setminus \{\alpha\}} (z - \beta)^{\sigma_{\max}^{(\beta)}} \left[N_{(\alpha)}^T \ D_{(\alpha)}^T \right]^T$ with $D_{(\alpha)} = I$ are q different solutions of equations (2.1) with $D_{(\alpha)}(\alpha)$ regular. Because these solutions have to be generated by the m basis elements, the theorem is proved. $\square$

Let us investigate now how we can use this fact to extract those elements from the submodule solving the two families of interpolation problems we have given above.

1. Let us retake the vector M-Padé approximation problem. If we set the shift parameters s_i equal to $s_i = -\alpha_i$ appearing in the degree conditions (2.2), we see that these conditions are equivalent to the condition that the $\vec{s}$ -degree has to be bounded by zero. Using our $\vec{s}$ -reduced basis, we can easily parametrize all polynomial m -tuples having $\vec{s}$ -degree ≤ 0 . This solves the problem.
2. To solve the second problem, we note that the rows of $[N^T \ D^T]$ can always be made $\vec{s}$ -reduced with $\vec{s}$ -degrees δ_i without changing the $\vec{s}$ -McMillan degree. In this case, the $\vec{s}$ -McMillan degree of the right coprime polynomial matrix fraction description ND^{-1} is equal to the sum of the $\vec{s}$ -degrees δ_i . Using our $\vec{s}$ -reduced basis and the result of Theorem 3.3, we can easily parametrize all solutions having minimal $\vec{s}$ -McMillan degree. This solves the second family of interpolation problems.

4 Computation of an $\vec{s}$ -reduced basis

To compute an $\vec{s}$ -reduced basis in a recursive way, we need the following theorem.

Theorem 4.1 *$\mathcal{B}$ is an $\vec{s}$ -reduced basis for the submodule S of Theorem 3.1, determined by the interpolation indices $\sigma_i^{(\alpha)}$ iff the m elements of $\mathcal{B}$ belong to S , are $\mathbb{F}[z]$ -linearly independent and the sum of their $\vec{s}$ -degrees is equal to $k_S + \sum_{i=1}^m s_i$ with k_S a constant integer independent of the shift parameters s_i but depending on the submodule S in the following way. If all interpolation indices are zero, the submodule $S = \mathbb{F}[z]^m$ and $k_S = 0$. If we increment one of the interpolation indices $\sigma_i^{(\alpha)}$, determining the submodule S , for a specific interpolation point α and a specific row i resulting in a submodule S' , there are only the following two possibilities:*

- all elements of S satisfy the additional interpolation condition; hence, $S' = S$, and $k_{S'} = k_S$,
- not all elements of S satisfy the additional interpolation condition; hence, $S' \subset S$, and $k_{S'} = k_S + 1$.

The number k_S is the number of independent equations of the set of linear homogeneous equations defining the submodule S . Hence, k_S is independent of the ordering of the interpolation indices.

Proof. ($\Rightarrow$) Let us consider the submodule S , determined by the interpolation indices $\sigma_i^{(\alpha)}$. From Theorem 3.1, we know that an $\vec{s}$ -reduced basis for S consists of exactly m polynomial m -tuples $\vec{b}_i$ with $\vec{s}$ -degree δ_i . Let $\delta_{\max} = \max\{\delta_i : i = 1, \dots, m\}$ and $s_{\max} = \max\{s_i : i = 1, \dots, m\}$. Consider $\delta \geq \max\{\delta_{\max}, s_{\max}\}$. We want to compute all polynomial m -tuples having $\vec{s}$ -degree $\leq \delta$ satisfying the interpolation conditions (2.1). We can consider the equations (2.1) as a set of linear homogeneous equations with $\sum_{i=1}^m (\delta + 1 - s_i)$ unknown coefficients. From Theorem 3.2, we easily derive that the number of free parameters is equal to $\sum_{i=1}^m (\delta + 1 - \delta_i)$. Therefore, the number of linear independent equations of the system is

$$\sum_{i=1}^m (\delta + 1 - s_i) - \sum_{i=1}^m (\delta + 1 - \delta_i) = \sum_{i=1}^m \delta_i - \sum_{i=1}^m s_i \quad (4.5)$$

which is independent of δ as long as δ is taken large enough. This number is also independent of the choice of the shift parameters s_i . Indeed, the contribution of the shift parameters s_i through the $\vec{s}$ -degrees δ_j is annihilated by subtracting the sum of the s_i . By induction, it is easy to prove that the number (4.5) is k_S . It is clear that incrementing one of the interpolation indices adds a new linear homogeneous equation. Hence, k_S changes into $k_{S'}$, following the rules described above.

($\Leftarrow$) It remains to be shown that when we take m $\mathbf{F}[z]$ -linearly independent elements $\vec{b}'_i$ of S with the sum of their $\vec{s}$ -degrees δ'_i equal to $k_S + \sum_{i=1}^m s_i$, then these elements form an $\vec{s}$ -reduced basis of S . Note that

$$\sum_{i=1}^m \delta'_i \leq \deg \det(B' D) = \deg \det B' + \sum_{i=1}^m s_i \quad (4.6)$$

with $B' = [\vec{b}'_1^T \dots \vec{b}'_m^T]^T$ and $D = \text{diag}(z^{s_1}, \dots, z^{s_m})$. We have equality iff the $\vec{b}'_i$ are $\vec{s}$ -reduced. Because there always exists a $\vec{s}$ -reduced basis for S , $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_m\}$, we can write $B' = UB$ with U an $(m \times m)$ polynomial matrix and B defined similar to B' . Therefore, $\deg \det B' = \deg \det U + \deg \det B$. Because by (4.6) $\deg \det B' \geq \sum_{i=1}^m \delta'_i - \sum_{i=1}^m s_i = k_S$, and $\deg \det B = \sum_{i=1}^m \delta_i - \sum_{i=1}^m s_i = k_S$, we easily derive that $k_S \leq \deg \det U + k_S$. Hence, $\deg \det U = 0$ or U is unimodular and we have to take equality in the previous formulas leading to the result of the theorem. $\square$

An $\vec{s}$ -reduced basis can be constructed in a recursive way as follows. First, note that the rows of the unit matrix $I_{m \times m}$ are $\vec{s}$ -reduced for any $\vec{s}$, thus forming a basis for the module $\mathbf{F}[z]^m$. Starting from this basis, we can recursively compute an $\vec{s}$ -reduced basis for any choice of shift parameters s_i , interpolation points $\alpha \in I$ and interpolation indices $\sigma_i^{(\alpha)}$ using the following three basic steps. Let us assume that we start with an $\vec{s}$ -reduced basis $\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m\}$ for the submodule S of Theorem 3.1.

1. We can increment one of the interpolation indices $\sigma_i^{(\alpha)}$ for a specific interpolation point α and a specific row i resulting in a submodule S' . This is equivalent to adding one linear homogeneous equation to the already existing linear set of homogeneous equations determined by equation (2.1). We shall write this linear homogeneous equation as

$$l(\vec{p}) = 0 \quad (4.7)$$

with $\vec{p}$ a polynomial m -tuple. However, if the polynomial m -tuple $\vec{p}$ does not satisfy equation (4.7), we shall get a nonzero right-hand side, which we call the residual $r_{\vec{p}}$ for $\vec{p}$. If $r_{\vec{b}_i} = 0, i = 1 \dots m$, $S' = S$ and we can keep the same $\vec{s}$ -reduced basis. In this case, $k_{S'} = k_S$. When at least one of the residuals $r_{\vec{b}_i} \neq 0$, we can use the following algorithm to determine an $\vec{s}$ -reduced basis for S' :

- take a basis element having nonzero residual and smallest $\vec{s}$ -degree; suppose this is the basis element $\vec{b}_l$;
- take the following $m - 1$ polynomial m -tuples

$$\vec{b}_i - r_{\vec{b}_i} / r_{\vec{b}_l} \vec{b}_l, \quad i = 1, \dots, m \text{ but } i \neq l;$$

- add the polynomial m -tuple $(z - \alpha)\vec{b}_l$ to get a set of m polynomial m -tuples.

Using Theorem 4.1, it is easy to show that this set is a possible choice of an $\vec{s}$ -reduced basis for S' taking into account that $k_{S'} = k_S + 1$ and the shift parameters s_j are not changed.

2. We can increment one of the shift parameters s_j giving more importance to the degree of the j -th component of the polynomial m -tuples in the global $\vec{s}$ -degree with $\vec{s}' = \vec{s} + \vec{e}_j$. Note that the interpolation conditions (2.1) do not change, i.e. the submodule S considered is the same, only the degree structure is different. Let us denote the j -th component of the $\vec{s}$ -highest degree coefficient of a polynomial m -tuple $\vec{p}$ as $H_j^{(\vec{s})}\vec{p}$. Using Theorem 4.1, it is easy to show that the following polynomial m -tuples form an $\vec{s}'$ -reduced basis for the same submodule S .

- take a basis element $\vec{b}_l$ having the smallest $\vec{s}$ -degree and $H_j^{(\vec{s})}\vec{b}_l \neq 0$;
- add the following $m - 1$ polynomial m -tuples

$$\vec{b}_i - z^{\delta_i - \delta_l} \left(H_j^{(\vec{s})}\vec{b}_i / H_j^{(\vec{s})}\vec{b}_l \right) \vec{b}_l, \quad i = 1, \dots, m \text{ but } i \neq l$$

with δ_i the $\vec{s}$ -degree of $\vec{b}_i$.

3. Similarly, we can decrement one of the shift parameters s_j . Let us denote the $\vec{s}$ -highest degree coefficient of a polynomial m -tuple $\vec{p}$ excluding the j -th component as $H_{\neq j}^{(\vec{s})}\vec{p}$. The set of linear equations

$$\sum_{i=1}^m c_i H_{\neq j}^{(\vec{s})}\vec{b}_i = \vec{0}$$

has a nontrivial solution $(c_1, c_2, \dots, c_m) \neq \vec{0}$. Using Theorem 4.1, it is easy to show that the following set of polynomial m -tuples forms an $\vec{s}'$ -reduced basis set for the same submodule S .

- look for the index l with $c_l \neq 0$ and δ_l as large as possible; take the polynomial m -tuple $\sum_{i=1}^m c_i z^{\delta_l - \delta_i} \vec{b}_i$ as the first member of the new basis;
- the other $m - 1$ members of the new basis are the elements of the old basis, excluding the l -th one.

When we combine the basis elements as rows of a square $(m \times m)$ polynomial matrix, each of these steps involves a left multiplication by a unimodular matrix, in case of incrementing one of the interpolation indices possibly followed by a multiplication of one of the rows of the polynomial matrix by $(z - \alpha)$. Each of these elementary steps is very simple, efficient and straightforward, and requires $O(mk_S)$ or $O(m \sum_{\alpha \in I} \sum_{i=1}^p \sigma_i^{(\alpha)})$ floating point operations at the maximum. These three basic steps allow to follow any “path” through the “solution table”. For example, considering the scalar linearized Padé approximation problem, one can follow a diagonal, a row, a column, an antidiagonal, ..., and any combination of these in the Padé table. For example, to go from one entry of a row in the Padé table to the next entry, one increments the interpolation index σ (connected to the interpolation point 0) and decrements s_2 (or equivalently increments s_1).

5 Conclusion

The proposed framework will be useful to solve certain interpolation problems in a very general form such that the solution to more specific classical problems can be easily derived from the general solution method. This framework will also allow us to investigate the connections between several of these interpolation problems and between the ways in which they are solved. For example, the method described by Beckermann [1] to solve the scalar M-Padé approximation problem (first family) is very similar to the one described in [8] to solve the vector rational interpolation problem (second family).

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