Optimized Overlapping Schwarz Methods for Parabolic PDEs with Time-Delay

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Summary. We present overlapping Schwarz methods for the numerical solution of two model problems of delay PDEs: the heat equation with a fixed delay term, and the heat equation with a distributed delay in the form of an integral over the past. We first analyze properties of the solutions of these PDEs and find that their dynamics is fundamentally different from that of regular time-dependent PDEs without time delay. We then introduce and study overlapping Schwarz methods of waveform relaxation type for the two model problems. These methods compute the local solution in each subdomain over many time-levels before exchanging interface information to neighboring subdomains. We analyze the effect of the overlap and derive optimized transmission conditions of Robin type. Finally we illustrate the theoretical results and convergence estimates with numerical experiments.

1 Introduction

Delay differential equations model physical systems for which the evolution does not only depend on the present state of the system but also on the past history. Such models are found, for example, in population dynamics and epidemiology, where the delay is due to a gestation or maturation period, or in numerical control, where the delay arises from the processing in the controller feedback loop. Delay differential equations have been studied extensively (and almost exclusively) in the context of ordinary differential equations. An ordinary delay differential equation is an equation of the form

$$\dot{y}(t) = F(t, y(t), y_t), \quad t \in [0, T], \tag{1}$$

where y_t denotes a function segment extending over a time-interval of length τ into the past: $y_t(s) = y(t+s)$, $s \in [-\tau, 0]$. Equation (1) is usually complemented with an initial condition of the type $y_0(s) = g(s)$, where $g(s)$ is a given function over the interval $s \in [-\tau, 0]$. A good starting point to study the analysis and numerical computation of ordinary delay differential equations is Bellen and Zennaro [2003], and the references therein.

Delay PDEs are less well understood. They are typically of the form

$$\frac{\partial}{\partial t}u(t,x) = \mathcal{L}u(t,x,u_{(t,x)}) + f(t,x), \tag{2}$$

where $u_{(t,x)}$ is a function segment, which can extend both in the past and over some region in space: $u_{(t,x)}(v,w) = u(t+v, x+w)$, $(v,w) \in [-\tau,0] \times [-\sigma,\sigma]$. Equation (2) has to be completed with boundary conditions and an initial condition, which, typically, have to be specified over some initial and boundary regions around the domain of definition of the delay PDE. A set of examples, illustrating the wide range of existing delay PDE models can be found in Wu [1996]. A characteristic example from numerical control is the equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + v(g(u(t-\tau,x)))\frac{\partial u}{\partial x} + c[f(u(t-\tau,x)) - u(t,x)],$$

which models a furnace used to process metal sheets. Here, u is the temperature distribution in a metal sheet, moving at a velocity v and heated by a source specified by the function f; both v and f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length τ. An example from population dynamics is the so-called Britton-model,

$$\frac{\partial u}{\partial t} = D\Delta u + u(1 - g \star u) \quad \text{with} \quad g \star u = \int_{t-\tau}^{t}\int_{\Omega} g(t-s, x-y)u(s,y)\,dy\,ds.$$

Here, $u(t,x)$ denotes a population density, which evolves through random migration (modeled by the diffusion term) and reproduction (modeled by the nonlinear reaction term). The latter involves a convolution operator with a kernel $g(t,x)$, which models the distributed age-structure dependence of the evolution and its dependence on the population levels in the neighborhood.

There is little experience with numerical methods for solving delay PDEs. Zubik-Kowal [2001] and Huang and Vandewalle [2003] analyze the accuracy and stability of spatial and temporal discretization schemes. Zubik-Kowal and Vandewalle [1999] analyze the convergence of a waveform relaxation scheme of Gauss-Seidel and Jacobi type, for solving the discretized problems. In this paper we present a first analysis of domain decomposition based waveform relaxation methods for the solution of two model delay PDEs. Waveform relaxation schemes using domain decomposition in space for parabolic equations without delay were introduced in Gander and Stuart [1998] and independently in Giladi and Keller [2002], and further analyzed in Gander [1998] and Gander and Zhao [2002], see also the references therein. In those papers, it was shown that domain decomposition leads to a fundamentally faster convergence rate than the classical waveform relaxation methods. The performance of these methods can however still be drastically improved using better transmission conditions between subdomains, see Gander et al. [1999]. Our goal is to analyse whether such optimization is also possible in the parabolic delay PDE case.

The structure of the paper is as follows. In §2 we define two characteristic models of delay PDEs, and we analyze the stability of the solution of those problems as a function of the parameters appearing in the model. In §3, we analyze the performance of the classical overlapping Schwarz waveform relaxation method when used as a solver for delay PDEs. An algorithm using optimized Robin type transmission conditions is studied in §4. Finally, in §5, the theoretical results are verified by some numerical experiments.

2 Analysis of Delay PDEs

We consider two representative model problems: a PDE with a constant delay and one with a distributed delay. By analyzing the properties of their solutions, we hope to gain some insight into the behavior of solutions to the more complex problems introduced in §1. The constant delay PDE is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au(t-\tau), \quad \text{with} \begin{cases} x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ a \in \mathbb{R},\ \tau \in \mathbb{R}^+. \end{cases} \tag{3}$$

Using separation of variables, we arrive at solutions of the form $u(t,x) = e^{\lambda t} \cdot e^{ikx}$. The constants $\lambda \in \mathbb{C}$ and $k \in \mathbb{R}$ satisfy the so-called characteristic equation $\lambda = -k^2 + ae^{-\lambda\tau}$. Separating real and imaginary parts, $\lambda = \eta + i\omega$, we obtain the system of equations

$$\begin{cases} \eta = -k^2 + ae^{-\eta\tau}\cos(\omega\tau), \\ \omega = \qquad\ - ae^{-\eta\tau}\sin(\omega\tau). \end{cases} \tag{4}$$

The natural question that arises therefore is for what (a,τ)-pairs the characteristic equation has only solutions with $\eta \leq 0$ and thus solutions of the delay PDE stay bounded for all time. To answer this question of stability, we distinguish two cases. First, we identify the region in the (a,τ) parameter space where unstable solutions exist corresponding to real roots λ; next we treat the unstable, oscillatory solutions case, i.e. corresponding to complex-valued roots with non-vanishing imaginary part ω.

Setting $\omega = 0$ and $\eta > 0$, equation (4) simplifies to $\eta = -k^2 + ae^{-\eta\tau}$. For positive a, and a given k, this equation has a unique solution η, as illustrated in Figure 1 (left). If $k^2 < a$, the corresponding η is positive. Hence, for any $a > 0$ there always exist modes (with k small enough) that grow exponentially. A similar graphical argument shows that, for $a < 0$, any roots η must necessarily be negative. Hence, there are no unstable real modes in that case.

Next, by setting $\omega > 0$ and $\eta = 0$ in (4) we determine the boundary of the (a,τ)-region where unstable oscillatory modes exist. This leads to the conditions $k^2 = a\cos(\omega\tau)$ and $-\omega = a\sin(\omega\tau)$. An analysis of these conditions reveals that they can only be satisfied for $a < 0$ if $\pm\,\omega\tau \in [\pi/2, \pi] + 2\pi n$ with n an arbitrary positive integer; for $a > 0$ the corresponding condition becomes $\pm\,\omega\tau \in [3\pi/2, 2\pi] + 2\pi n$.

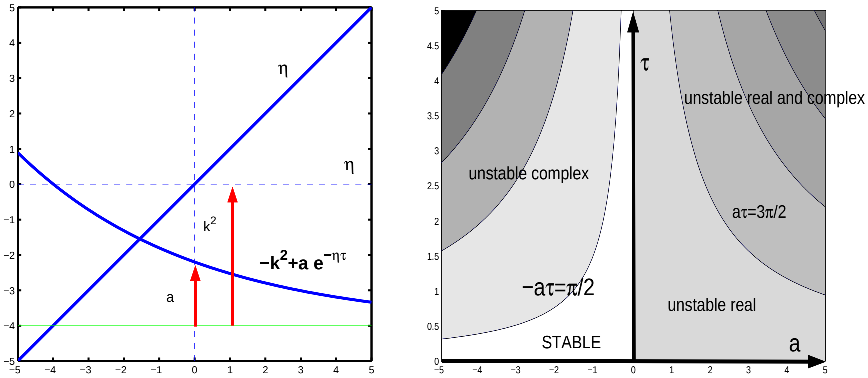

Fig. 1. Left: stability analysis of the constant delay PDE, the case of real roots. Right: stable and unstable regions in the (a, τ) parameter space.

The curves $a\tau = -\omega\tau / \sin(\omega\tau)$, for $\omega\tau = \pi/2 + \pi n$ are especially important. They determine the (a, τ)-values at which the constant mode, $k=0$, becomes unstable, with an oscillation determined by the corresponding ω. It can be shown that the constant mode is the first mode to become unstable; this leads to the theorem below. In Figure 1 (right) the stability region is shown in white. In the other regions each unstable mode is of a specific multiplicity.

Theorem 1. *The solution to the constant delay PDE* (3) *is stable if* $-\pi/2 \leq a\tau \leq 0$*.*

The second model problem is a distributed delay PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a \int_{-\tau}^{0} u(t+s)\, ds \quad \text{with} \quad \begin{cases} x \in \mathbb{R},\, t \in \mathbb{R}^+, \\ a \in \mathbb{R},\, \tau \in \mathbb{R}^+. \end{cases} \tag{5}$$

Now, the characteristic equation is given by $\lambda = -k^2 + \frac{a}{\lambda}(1 - e^{-\lambda\tau})$. Separating real and imaginary parts as we did before, one obtains the system

$$\begin{cases} \eta^2 - \omega^2 + \eta k^2 = a(1 - \cos(\omega\tau)e^{-\eta\tau}), \\ 2\eta\omega + \omega k^2 \qquad = a \sin(\omega\tau)e^{-\eta\tau}. \end{cases} \tag{6}$$

We determine the (a, τ)-pairs for which the characteristic equation has only solutions with $\eta \leq 0$. An elementary graphical argument reveals that any positive a admits unstable real roots, i.e., with $\omega = 0$. There are no such roots for $a < 0$. Setting $\eta = 0$ in (6), we arrive at two equations, which can only be satisfied for $a < 0$ and for $\pm\, \omega\tau \in [\pi, 2\pi] + 2\pi n$. As before, the constant mode, with $k=0$, is the stability determining one. The curves $a\tau^2 = -\omega^2\tau^2/(1 - \cos(\omega\tau))$ for $\omega\tau = \pi + 2\pi n$ determine the (a, τ)-values where a constant mode instability appears. Figure 2 shows the stability region.

Theorem 2. *The solution to the distributed delay PDE* (5) *is stable if* $-\pi^2/2 \leq a\tau^2 \leq 0$*.*

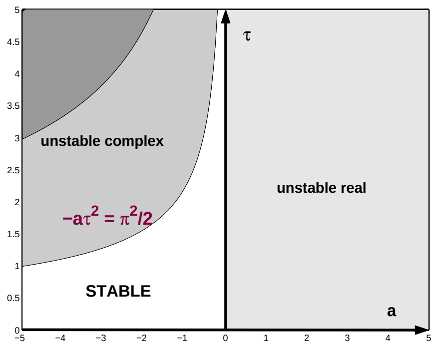

Fig. 2. Stable and unstable regions for the distributed delay PDE.

3 Domain Decomposition

The classical Schwarz algorithm. We decompose the domain of PDE (3) into two overlapping subdomains $\Omega_1 = (-\infty, L)$ and $\Omega_2 = (0, \infty)$, with overlap $L > 0$. The classical Schwarz waveform iteration is then given by

$$\begin{cases} \frac{\partial u_1^n}{\partial t} = \frac{\partial^2 u_1^n}{\partial x^2} + a u_1^n(t-\tau) \ \text{ on } \Omega_1, \ u_1^n(t,L) = u_2^{n-1}(t,L), \\ \frac{\partial u_2^n}{\partial t} = \frac{\partial^2 u_2^n}{\partial x^2} + a u_2^n(t-\tau) \ \text{ on } \Omega_2, \ u_2^n(t,0) = u_1^{n-1}(t,0), \end{cases} \tag{7}$$

starting with some initial guesses $u_1^0(t,L)$ and $u_2^0(t,0)$. For the analysis we will assume those to be in L^2. Using Laplace transforms, we can rewrite (7) as an iteration in 'frequency space' and explicitly solve for the Laplace transform of the solutions $u_1^n(t,L)$ and $u_2^n(t,0)$. Using arguments very similar to the ones in Gander et al. [1999] we arrive at the following result.

Theorem 3. *Assume a and τ satisfy the stability condition of Theorem 1. Then, the classical Schwarz waveform relaxation algorithm (7) for the constant delay PDE (3) converges linearly, i.e., with $e_1^n = u - u_1^n$ and $e_2^n = u - u_2^n$,*

$$||e_1^n(\cdot,L)||_2 + ||(e_2^n(\cdot,0)||_2 \leq \rho^n(||e_1^0(\cdot,L)||_2 + ||e_2^0(\cdot,0)||_2), \tag{8}$$

where $\rho := \rho_{cla} = \sup_{\omega\in\mathbb{R}} \left| \, e^{-\sqrt{i\omega - ae^{-i\omega\tau}}\,L} \, \right| < 1.$

The full details of the derivation are given in the companion report Vandewalle and Gander. Using elementary, but very technical arguments, the convergence rate ρ_{cla} can be bounded, as a function of the problem parameters and the size of the overlap.

Corollary 1. *The convergence rate of the classical Schwarz method for problem* (3) *satisfies* $\rho_{cla} \leq e^{-\sqrt{-a\cos(a\tau)/2}\,L}$, *provided* $-a\tau < 1$.

Next, we consider the Schwarz algorithm for the distributed delay PDE,

$$\begin{cases} \frac{\partial u_1^n}{\partial t} = \frac{\partial^2 u_1^n}{\partial x^2} + a\int_{-\tau}^0 u_1(t+s)\,ds & \text{on } \Omega_1, \quad u_1^n(t,L) = u_2^{n-1}(t,L), \\ \frac{\partial u_2^n}{\partial t} = \frac{\partial^2 u_2^n}{\partial x^2} + a\int_{-\tau}^0 u_2(t+s)\,ds & \text{on } \Omega_2, \quad u_2^n(t,0) = u_1^{n-1}(t,0). \end{cases} \tag{9}$$

With a Laplace transform analysis similar to that for the constant delay case, and some technical arguments, we derive the following theorem and corollary.

Theorem 4. *Assume a and τ satisfy the stability condition of Theorem 2. Then, the classical Schwarz algorithm (9) for delay PDE (5) converges linearly as in (8), where* $\rho := \rho_{cla} = \sup_{\omega\in\mathbb{R}} \left| e^{-\sqrt{i\omega + i\frac{a}{\omega}(1-e^{-i\omega\tau})}\,L} \right| < 1.$

Corollary 2. *The convergence rate of the classical Schwarz method for problem* (5) *satisfies* $\rho \le e^{-\sqrt{\sqrt{-a}\sin(\sqrt{-2a\tau})/2}\,L}$.

The optimized Schwarz algorithm. We introduce new transmission conditions in (7) and (9), using $\mathcal{B}_+$ and $\mathcal{B}_-$ to denote $(\frac{\partial}{\partial x}+p)$ and $(\frac{\partial}{\partial x}-p)$,

$$\mathcal{B}_+ u_1^n(t,L) = \mathcal{B}_+ u_2^{n-1}(t,L), \quad \mathcal{B}_- u_2^n(t,0) = \mathcal{B}_- u_1^{n-1}(t,0). \tag{10}$$

Theorem 5. *Assume a and τ satisfy the stability condition of Theorem 1. Then, the Schwarz waveform relaxation algorithm (7) with the Robin transmission conditions (10) converges as stated in (8), where*

$$\rho := \rho_{opt}(p) = \sup_{\omega\in\mathbb{R}} \left| \frac{\sqrt{i\omega - ae^{-i\omega\tau}} - p}{\sqrt{i\omega - ae^{-i\omega\tau}} + p} e^{-\sqrt{i\omega - ae^{-i\omega\tau}}\,L} \right| < 1. \tag{11}$$

Defining the curve $\Gamma = \{z : z = \sqrt{i\omega - ae^{-i\omega\tau}}, \omega\in\mathbb{R}\}$, the optimal choice of the parameter p is the value $p^\star$ which solves the min-max problem

$$\min_p \max_{z\in\Gamma} \left| \frac{z-p}{z+p} \cdot e^{-z\,L} \right|. \tag{12}$$

In Figure 3 we graphically depict the curve Γ, together with the contour lines of the function that appears in the min-max problem, for the case $L=0$.

In a numerical computation, $\omega \in [-\omega_{\max}, \omega_{\max}]$, because a numerical grid in time with spacing Δt can not carry arbitrary high frequencies; an estimate of $\omega_{\max}$ is $\omega_{\max} = \frac{\pi}{\Delta t}$. This simplifies the min-max problem (12) to a problem in a bounded domain, but it is still difficult to solve analytically, even for the special case $L = 0$. We therefore propose to solve the min-max problem over the bounding box given in Figure 3 containing the curve. This problem can be solved in closed form for $L = 0$.

Theorem 6. *Let $L = 0$ and set $b = \Re(\sqrt{i(\omega_{\max} + 2\pi/\tau) - ae^{-i\omega_{\max}\tau}})$. Assume $-a\tau \le 1$. If $b \ge -a\cos(a\tau) + 1/\cos(a\tau)$, then the solution of the approximate min-max problem is given by $p^\star = \sqrt{2\cos(a\tau)b + a}$; otherwise it is $p^\star = \sqrt{2a^2\cos^2(a\tau) - a}$.*

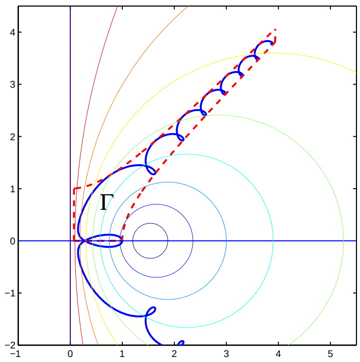

Fig. 3. Curve Γ along which one needs to solve the min-max problem to find the optimal p in the Robin transmission conditions, and level sets of $|z-\frac{3}{2}|/|z+\frac{3}{2}|$.

The parameter $p^\star$ guarantees a lower bound on achievable acceleration,

$$\rho_{opt}(p^\star) \leq \sqrt{\frac{(p^\star-q)^2+q^2-a}{(p^\star+q)^2+q^2-a}} \cdot \rho_{cla}, \quad q=-a\cos(a\tau). \tag{13}$$

A similar analysis can also be done for the distributed delay PDE, leading to a min-max problem as in (12), along a curve

$$\tilde{\Gamma} = \{z : z = \sqrt{i\omega + i\frac{a}{\omega}(1-e^{-i\omega\tau})}, \omega \in \mathbb{R}\}. \tag{14}$$

One can show that $\tilde{\Gamma}$ belongs to the same bounding box as the curve Γ. Hence, using the value of $p^\star$ from Theorem 6, a similar convergence acceleration will be achieved over the classical algorithm as in (13).

4 Numerical Results

We investigate the influence of the overlap L on the convergence of the classical Schwarz algorithm, and the influence of the parameter p in the Robin transmission conditions, on the convergence of the optimized Schwarz method. The results presented are for the constant delay PDE. We chose the parameters $a=-1.55$, $\tau=1$, i.e., within the stability region, and $x \in [0,2]$, $t \in [0,10]$, $\Delta x = \frac{1}{50}$ and $\Delta t = \frac{1}{50}$. In Figure 4 (left) we show the evolution of the error as a function of the iteration index n, for various values of the overlap. The convergence improvement with increasing overlap is evident. The influence of the parameter in the Robin transmission conditions is shown in Figure 4 (right). Here, a minimal overlap of size $L=\Delta x$ was used.

Our experiments show clearly that the transmission conditions play a very important role for the performance of the algorithm. Compared to the overlap, where an increase corresponds to an increase in the subdomain solution cost, a change in p does not increase the subdomain solution cost.

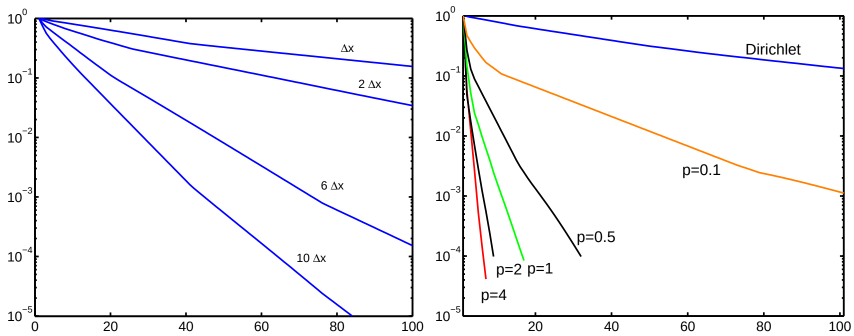

Fig. 4. Left: influence of the overlap on the performance of the classical Schwarz waveform relaxation algorithm for the constant delay PDE problem. Right: influence of the parameter p in the Robin transmission condition on the performance of the optimized Schwarz algorithm.

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