ON THE CONVERGENCE OF SCHUR PARAMETERS FOR A TOEPLITZ MATRIX WITH A MEROMORPHIC SYMBOL

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The limit behavior of Schur parameters is investigated when they are associated to Toeplitz matrices of growing size whose entries are the Laurent coefficients of a meromorphic function.

1. Introduction

In his celebrated paper [23], I. Schur studies a class of functions that are now generally known as Schur functions. A Schur function is an analytic function with a modulus not exceeding unity in the open unit disc. Most remarkable is that such functions can be characterized by a sequence of parameters that can be obtained from its Fourier coefficients by some elegant algorithm.

These Schur parameters and the algorithm constructing them are known to be related to many other problems.

It was pointed out by Akhiezer [1] that the Schur parameters are exactly those occurring in the recurrence relations on Szegő's orthogonal polynomials [25] associated with a weight function that is somehow linked with the underlying Schur function.

Szegő's polynomials and therefore also the Schur parameters appear in several engineering applications. It is e.g. shown in [7] by P.Dewilde et al that the Szegő recurrence relation can be recovered from a Darlington synthesis procedure known in circuit theory. The techniques used in [7] are those from inverse scattering theory [21].

Applications in linear estimation theory for stochastic processes and digital filtering are also well known [11]. E.g. the construction of an autoregressive predictor involves the computation of polynomials orthogonal on the unit circle with respect to the density function of the given stochastic process [11]. In that respect, the well known Levinson algorithm is worth mentioning [17]. The construction of a digital filter is essentially the same problem [18]. The Levinson algorithm computes the predictor polynomial from the covariance information of the stationary stochastic process. This matrix is a positive definite Toeplitz matrix. By this algorithm we obtain in fact a triangular factorization for the inverse of this matrix, while the algorithm of Schur can be seen as a dual algorithm since it can be formulated such that it gives the triangular factors of the covariance matrix itself [4].

The algorithms of Levinson and Schur attracted the attention of people from linear algebra since they give order n^2 algorithms to find the triangular factors of a Toeplitz matrix, hence to solve a linear system of equations with a Toeplitz matrix. These algorithms are called fast algorithms in comparison with general system solvers like Gauss elimination which is an order n^3 algorithm. In linear prediction applications, these Toeplitz matrices are positive definite and selfadjoint. It should be emphasised that these properties are not essential to construct an order n^2 algorithm. The reason why fast algorithms do exist for these problems is the low displacement rank property of Toeplitz matrices [14]. Trench [26,27], Zohar [28,29] and later many others developed fast algorithms to solve a general Toeplitz or Hankel set of linear equations, and even to solve more general systems with a matrix that is said to be close to Toeplitz [14]. These algorithms are in most cases generalizations of the Levinson and Schur algorithms. A lot of information on these fast algorithms with their applications to linear dynamical systems, signal processing and their VLSI implementation are found in [16]. [12] is another interesting reference on

these algorithms.

For a nonselfadjoint Toeplitz matrix, the original algorithms of Levinson and Schur have to be slightly adapted. Due to the asymmetry, we have now for each step in the recurrence relation of the algorithm two "Schur parameters" that are complex conjugate in the selfadjoint case. On the other hand, because the matrix is not positive definite in general, these parameters lose their typical Schur property of being bounded by one in modulus. We could call them generalized Schur parameters, but in this text we shall call them Schur parameters for short.

There are several problems where systems of linear equations with a general Toeplitz matrix have to be solved. E.g. discrete Wiener-Hopf equations when solved by the projection method involve the solution of nested sets of Toeplitz systems with an increasing dimension [8]. Exactly the same situation occurs when computing Padé approximations with constant numerator degree and increasing denominator degree [4]. More general equations with a Laurent matrix are also considered in [8]. These are equivalent with a problem of Laurent-Padé approximation [10].

In situations like in the latter, one is interested in what happens if the dimension of the system grows to infinity. Do the solutions of the systems converge? What happens to the Schur parameters? etc.

It is the subject of this paper to say something on these problems when the entries in the Toeplitz matrix are the coefficients of a Laurent series which converges to some meromorphic function in an annular region centered around the origin. Such problems have been considered when the Toeplitz matrix is generated from the Fourier coefficients of a positive definite function on the unit circle [11]. Our problem is at the same time more general since we use Laurent coefficients of a function which need not be positive, but also more restrictive because we suppose the function is meromorphic.

It will turn out that from studying the asymptotics of the Schur parameters, we shall obtain information about the location of the zeros of this function. This is not without importance for the positive definite case because we shall be able to find the zeros of a positive definite function by considering Toeplitz systems based on shifted Fourier coefficients. This zero information may then be used to construct predictors of more general structure than the autoregressive ones produced by the Schur algorithm. They are of so called autoregressive-moving average type [18]. They can be obtained by an algorithm developed by Nevanlinna and Pick [19,20] which is a rational interpolation algorithm generalizing the Schur algorithm in the sense that if all the interpolation points coincide at the origin, then the Nevanlinna-Pick algorithm reduces to the Schur algorithm.

2. Schur parameters and algorithms of Levinson and Schur type

Let $F(z) = \sum_{k=-\infty}^{\infty} f_k z^k$ be a formal complex Laurent series. With $F(z)$ we associate the infinite Toeplitz matrix

$$T_{\infty}^{(0)} = (f_{i-j})_{i,j=0}^{\infty}.$$

We call $F(z)$ the symbol of this Toeplitz matrix. It will turn out to be useful to consider also Toeplitz matrices with symbols $z^{-m} F(z)$ for $m \in \mathbb{Z}$. These will be denoted by

$$T_{\infty}^{(m)} = (f_{m+i-j})_{i,j=0}^{\infty}.$$

For further reference we shall give the development for general m .

Suppose the principal submatrices $T_n^{(m)} = (f_{m+i-j})_{i,j=0}^n$, $n = 0, 1, 2, \dots$ are all nonsingular. Then the systems

$$T_n^{(m)} Q_n^{(m)} = [v_n^{(m)} 0 \dots 0]^T \text{ and } T_n^{(m)} \hat{Q}_n^{(m)} = [0 \dots 0 \hat{v}_n^{(m)}]^T \quad (2.1)$$

with

$$Q_n^{(m)} = [q_{0,n}^{(m)} \dots q_{n,n}^{(m)}], q_{0,n}^{(m)} = 1$$

$$\hat{Q}_n^{(m)} = [\hat{q}_{0,n}^{(m)} \dots \hat{q}_{n,n}^{(m)}], \hat{q}_{n,n}^{(m)} = 1$$

have a unique solution and $v_n^{(m)} \hat{v}_n^{(m)} \neq 0$. The solutions of these systems satisfy a coupled recurrence relation:

$$Q_{n+1}^{(m)} = \begin{bmatrix} Q_n^{(m)} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{Q}_n^{(m)} \end{bmatrix} \alpha_{n+1}^{(m)} \quad (2.2a)$$

$$\hat{Q}_{n+1}^{(m)} = \begin{bmatrix} 0 \\ \hat{Q}_n^{(m)} \end{bmatrix} - \begin{bmatrix} Q_n^{(m)} \\ 0 \end{bmatrix} \hat{\alpha}_{n+1}^{(m)} \quad (2.2b)$$

with

$$\alpha_{n+1}^{(m)} = u_n^{(m)} / v_n^{(m)} \text{ and } \hat{\alpha}_{n+1}^{(m)} = \hat{u}_n^{(m)} / \hat{v}_n^{(m)}, n \geq 0 \quad (2.3)$$

$$u_n^{(m)} = [f_{m+n+1} \dots f_{m+1}] Q_n^{(m)}, \hat{u}_n^{(m)} = [f_{m-1} \dots f_{m-n-1}] \hat{Q}_n^{(m)} \quad (2.4)$$

and

$$v_{n+1}^{(m)} = \hat{v}_{n+1}^{(m)} = v_n^{(m)} (1 - \alpha_{n+1}^{(m)} \hat{\alpha}_{n+1}^{(m)}). \quad (2.5)$$

The initial conditions are

$$Q_0^{(m)} = \hat{Q}_0^{(m)} = 1 \text{ and } v_0^{(m)} = \hat{v}_0^{(m)} = f_m.$$

The recurrence is easily verified by filling it in in the defining relations (2.1). Some readers will recognize in these relations the algorithm developed by Trench and Zohar [26-29] to invert a finite Toeplitz matrix by the bordering technique. If we introduce the polynomials $Q_n^{(m)}(z) = [1 \ z \ \dots \ z^n] Q_n^{(m)}$ and $\hat{Q}_n^{(m)}(z) = [1 \ z \ \dots \ z^n] \hat{Q}_n^{(m)}$, then the recurrence (2.2) can be rewritten in the compact form

$$[Q_{n+1}^{(m)}(z) \hat{Q}_{n+1}^{(m)}(z)] = [Q_n^{(m)}(z) \hat{Q}_n^{(m)}(z)] \theta_{n+1}^{(m)} \quad (2.6)$$

$$\text{with } \theta_{n+1}^{(m)} = \begin{bmatrix} 1 & -\hat{\alpha}_{n+1}^{(m)} \\ -z \alpha_{n+1}^{(m)} & z \end{bmatrix}. \quad (2.7)$$

If $\{f_k\}_{-\infty}^{\infty}$ are the Fourier coefficients of some function $F(z)$, positive on the unit circle, then $f_{-k} = \bar{f}_k$ (upper bar stands for complex conjugate) and it follows from their definitions that $\hat{Q}_n^{(0)}(z) = z^n \overline{Q_n^{(0)}(1/\bar{z})}$ and $\hat{\alpha}_n^{(0)} = \overline{\alpha_n^{(0)}}$. The relations (2.1) express for $m = 0$ the orthogonality of $\hat{Q}_n^{(0)}(z)$ to z^k , $k = 0, 1, 2, \dots, n-1$ for the inner product

$$\langle p, q \rangle_F = \int_{-\pi}^{\pi} \overline{p(e^{i\theta})} F(e^{i\theta}) q(e^{i\theta}) d\theta. \quad (2.8)$$

Such polynomials are known as Szegő's polynomials [25] and (2.6-7) reduces in this special case to Szegő's recurrence relation. Since $\hat{v}_n^{(0)} = \|\hat{Q}_n^{(0)}\|^2 > 0$, it follows from (2.5) that the parameters for the Szegő case satisfy $|\alpha_n^{(0)}| < 1$. The previous algorithm for this special case will be recognized as the Levinson-Durbin algorithm which is basic in linear estimation theory. The coefficients of $Q_n^{(0)}(z)$ are then the predictor coefficients.

To develop the Schur type algorithm for computing the parameters $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$, we have to evaluate the formal products

$$F(z) Q_n^{(m)}(z) = \sum_{k=-\infty}^{\infty} h_k z^k, \quad h_k = \sum_{i=0}^n f_{k-i} q_{i,n}^{(m)} \quad (2.9a)$$

and

$$F(z) \hat{Q}_n^{(m)}(z) = \sum_{k=-\infty}^{\infty} \hat{h}_k z^k, \quad \hat{h}_k = \sum_{i=0}^n f_{k-i} \hat{q}_{i,n}^{(m)}. \quad (2.9b)$$

Both series are split up as follows

$$P_n^{(m)}(z) = \sum_{-\infty}^m h_k z^k, \quad R_n^{(m)}(z) = \sum_{m+1}^{\infty} h_k z^k \quad (2.10a)$$

and

$$\hat{P}_n^{(m)}(z) = \sum_{-\infty}^{m-1} \hat{h}_k z^k, \quad \hat{R}_n^{(m)}(z) = \sum_m^{\infty} \hat{h}_k z^k. \quad (2.10b)$$

From the definitions it follows that the leading coefficients in these series are precisely the numbers $u_n^{(m)}, \hat{u}_n^{(m)}, v_n^{(m)}$ and $\hat{v}_n^{(m)}$, used in the construction of $\alpha_{n+1}^{(m)}$ and $\hat{\alpha}_{n+1}^{(m)}$. Indeed

$$h_m = v_n^{(m)}, h_{m+1} = h_{m+2} = \dots = h_{m+n} = 0, h_{m+n+1} = u_n^{(m)} \quad (2.11a)$$

and

$$\hat{h}_{m-1} = \hat{u}_n^{(m)}, \hat{h}_m = \hat{h}_{m+1} = \dots = \hat{h}_{m+n-1} = 0, \hat{h}_{m+n} = \hat{v}_n^{(m)}. \quad (2.11b)$$

Introduce the recurrence relation (2.6-7) into

$$F(z)Q_n^{(m)}(z) = P_n^{(m)}(z) + R_n^{(m)}(z) \quad (2.12a)$$

and

$$F(z)\hat{Q}_n^{(m)}(z) = \hat{P}_n^{(m)}(z) + \hat{R}_n^{(m)}(z) \quad (2.12b)$$

and you find that also the formal series $P_n^{(m)}(z)$ and $\hat{P}_n^{(m)}(z)$ as well as the series $R_n^{(m)}(z)$ and $\hat{R}_n^{(m)}(z)$ satisfy the same recurrence relation (2.6-7). Thus

$$[S_{n+1}^{(m)}(z) \hat{S}_{n+1}^{(m)}(z)] = [S_n^{(m)}(z) \hat{S}_n^{(m)}(z)] \theta_{n+1}^{(m)} \quad (2.13)$$

with $\theta_{n+1}^{(m)}$ as in (2.7) and S one of Q, P or R .

This provides us with a Schur type algorithm to find the parameters $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$ for $n = 1, 2, \dots$. Starting with

$$P_0^{(m)}(z) = \sum_{-\infty}^m f_k z^k, \quad R_0^{(m)}(z) = \sum_{m+1}^{\infty} f_k z^k,$$

$$\hat{P}_0^{(m)}(z) = \sum_{-\infty}^{m-1} f_k z^k \quad \text{and} \quad \hat{R}_0^{(m)}(z) = \sum_m^{\infty} f_k z^k,$$

we find for $n = 1, 2, \dots$ the coefficients $u_n^{(m)}$, $\hat{u}_n^{(m)}$, $v_n^{(m)}$ and $\hat{v}_n^{(m)}$ as the leading coefficients in $R_n^{(m)}(z)$, $\hat{P}_n^{(m)}(z)$, $P_n^{(m)}(z)$ and $\hat{R}_n^{(m)}(z)$. With them find $\alpha_{n+1}^{(m)}$ and $\hat{\alpha}_{n+1}^{(m)}$ by (2.3) and use (2.13) to get $R_{n+1}^{(m)}(z)$, $\hat{R}_{n+1}^{(m)}(z)$, $P_{n+1}^{(m)}(z)$ and $\hat{P}_{n+1}^{(m)}(z)$.

The original algorithm of Schur is closer to a nonhomogeneous formulation which is found after introducing the formal quotients

$$\Gamma_n^{(m)}(z) = R_n^{(m)}(z)/(z\hat{R}_n^{(m)}(z)) = \sum_{k=0}^{\infty} \gamma_k z^k \quad (2.14a)$$

and

$$\hat{\Gamma}_n^{(m)}(z) = z\hat{P}_n^{(m)}(z)/P_n^{(m)}(z) = \sum_{k=0}^{\infty} \hat{\gamma}_k z^{-k}. \quad (2.14b)$$

It should be clear from the foregoing that $\gamma_0 = \alpha_{n+1}^{(m)}$ and $\hat{\gamma}_0 = \hat{\alpha}_{n+1}^{(m)}$, while recurrence (2.13) reduces to a Moebius transformation for $\Gamma_n^{(m)}(z)$ and $\hat{\Gamma}_n^{(m)}(z)$.

$$\Gamma_{n+1}^{(m)}(z) = z^{-1}t_{n+1}^{(m)}(\Gamma_n^{(m)}(z)) \quad \text{and} \quad \hat{\Gamma}_{n+1}^{(m)}(z) = z\hat{t}_{n+1}^{(m)}(\hat{\Gamma}_n^{(m)}(z)) \quad (2.15)$$

with

$$t_{n+1}^{(m)}(w) = \frac{w - \alpha_{n+1}^{(m)}}{1 - \hat{\alpha}_{n+1}^{(m)}w} \quad \text{and} \quad \hat{t}_{n+1}^{(m)}(w) = \frac{w - \hat{\alpha}_{n+1}^{(m)}}{1 - \alpha_{n+1}^{(m)}w}. \quad (2.16)$$

If $F(z)$ is a weight function as in the Szegő case and $\{f_k\}_{-\infty}^{\infty}$ its Fourier coefficients, then we have for $m = 0$ that due to the symmetry $\hat{\gamma}_k = \bar{\gamma}_k$, so that $\hat{\Gamma}_n^{(0)}(z)$ becomes superfluous. The algorithm as described in (2.15-16)

with $\alpha_{n+1}^{(0)} = \Gamma_n^{(0)}(0)$ and $\hat{\alpha}_{n+1}^{(0)} = \overline{\alpha_{n+1}^{(0)}}$ is precisely Schur's algorithm. Since we saw before that $|\alpha_n^{(0)}| < 1$, Schur's theorem says that $\Gamma_0^{(0)}(z)$ is a Schur function, i.e. it is analytic in the open unit disc and bounded by 1 in modulus. The parameters $\alpha_n^{(0)}$ are called Schur parameters. Since our parameters $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$ are their straightforward generalizations in a more general situation, we shall call them Schur parameters also.

3. Determinant expressions

Suppose for a moment that all the matrices $T_n^{(m)}$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$ are nonsingular. $F(z)$ is then called a normal series. For a normal series it follows from (2.1) and

$$T_n^{(m+1)} Q_n^{(m)} = [0 \cdots 0 \ u_n^{(m)}]^T \text{ and } T_n^{(m-1)} \hat{Q}_n^{(m)} = [\hat{u}_n^{(m)} \ 0 \cdots 0]^T \quad (3.1)$$

that all the $u_n^{(m)}$, $\hat{u}_n^{(m)}$, $v_n^{(m)}$ and $\hat{v}_n^{(m)}$ are nonzero. Hence all the Schur parameters exist and are nonzero. The subject of this paper is to find out what happens to the Schur parameters when $F(z)$ is a Laurent series converging to a meromorphic function in an annular region centered around the origin. The problem is that such a series need not be normal in general. This complicates the problem since not all the Schur parameters need to exist and they can not be computed by the recurrence algorithms introduced before. It is interesting to have closed form expressions for the Schur parameters as well as for the polynomials and series introduced before. These are gathered in the following theorem.

THEOREM 3.1 Whenever $\det T_{n-1}^{(m)} \neq 0$, the Schur parameters are given by

$$\alpha_n^{(m)} = (-1)^{n-1} \det T_{n-1}^{(m+1)} / \det T_{n-1}^{(m)} \quad (3.2a)$$

and

$$\hat{\alpha}_n^{(m)} = (-1)^{n-1} \det T_{n-1}^{(m-1)} / \det T_{n-1}^{(m)}. \quad (3.2b)$$

Under the same condition

$$S_n^{(m)}(z) = \det \begin{bmatrix} S_0^{(m+n)}(z) & zS_0^{(m+n-1)}(z) & \cdots & z^n S_0^{(m)}(z) \\ f_{m+1} & & & \\ \vdots & & T_{n-1}^{(m)} & \\ f_{m+n} & & & \end{bmatrix} / \det T_{n-1}^{(m)} \quad (3.3a)$$

and

$$\hat{S}_n^{(m)}(z) = \det \begin{bmatrix} & & f_{m-n} \\ & T_{n-1}^{(m)} & \vdots \\ & & f_{m-1} \\ \hat{S}_0^{(m+n)}(z) & z\hat{S}_0^{(m+n-1)}(z) & \cdots & z^n \hat{S}_0^{(m)}(z) \end{bmatrix} / \det T_{n-1}^{(m)} \quad (3.3b)$$

where S is one of Q, P or R and $S_n^{(m)}(z)$ and $\hat{S}_n^{(m)}(z)$ are defined in (2.1) and (2.10).

Proof: (2.6-7) shows that $q_n^{(m)}$, the last coefficient of $Q_n^{(m)}(z)$ is $-\alpha_n^{(m)}$. By Cramer's rule we find from the systems (2.1) that relation (3.2a) is true. To show (3.3a) for $S = Q$, note that $Q_0^{(k)}(z) = 1$ for all k , so that an expansion of (3.3a) along the first row gives precisely the coefficients as they would be found by Cramer's rule from the system (2.1). For $S = P$, we start from (3.3a) with $S = Q$. Multiply this with $F(z)$ and delete the terms in z^k for $k > m$. This gives $P_n^{(m)}(z)$ for the left hand side. In the right hand side, the multiplication is performed in the first row which becomes after the deletion of terms in $z^k, k > m$

$$\left[\sum_{-\infty}^m f_k z^k \quad z \sum_{-\infty}^{m-1} f_k z^k \quad \cdots \quad z^n \sum_{-\infty}^{m-n} f_k z^k \right].$$

Subtract from this z^{m+1} times the second row and z^{m+2} times the third row etc. and you obtain the result. The proof for $S = R$ is similar. For the quantities with a hat, the same technique is used.

□

Theorem 3.1 shows that the quantities exist when $\det T_n^{(m)} \neq 0$. Since we are interested in their behavior for n going to infinity, we should try to find out when these Toeplitz determinants are asymptotically nonzero. This is done in the next section.

4. Meromorphic functions and their Toeplitz determinants

Asymptotic expressions for Toeplitz determinants have been the subject of much research. A recent survey is given in [3]. We are particularly interested in asymptotic expressions that contain the zeros of the meromorphic symbol. Under certain conditions, such an expression was given by Day [6] for a rational symbol. We give here similar results for the meromorphic case as derived in [5].

Let $F(z)$ be the Laurent series, converging in an annular region $A = \{z: r < |z| < R\}$ to a function which is analytically continuable to a meromorphic function in $D = \{z: 0 < |z| < \infty\}$. The function will also be denoted by $F(z)$. We suppose A is restricted such that it contains no zeros of $F(z)$. The zeros ζ_k , $-Z^- \leq k \leq Z^+$ and the poles π_k , $-P^- \leq k \leq P^+$ of $F(z)$ are supposed to be numbered such that

$$0 \leq \dots \leq |\pi_{-1}| \leq |\pi_0| \leq r < \rho < R \leq |\pi_1| \leq |\pi_2| \leq \dots \leq \infty \quad (4.1)$$

$$0 \leq \dots \leq |\zeta_{-1}| \leq |\zeta_0| \leq r < \rho < R \leq |\zeta_1| \leq |\zeta_2| \leq \dots \leq \infty. \quad (4.2)$$

Each of these is repeated as many times as indicated by its multiplicity.

If Z^+ is infinite, then ∞ is an accumulation point of the $|\zeta_k|$, but every ζ_k with finite k is finite. If Z^+ is finite, we introduce the artificial number ζ_{Z^++1} such that $\max(R, |\zeta_{Z^+}|) < \zeta_{Z^++1} < \infty$. Similar observations are valid for Z^- , P^+ and P^- .

Any such function can be written in the form (see [8] chap. I sect.5)

$$F(z) = G_\rho(\tilde{F})F_+(z)F_-(z)z^\kappa. \quad (4.3)$$

For some number ρ such that $r < \rho < R$ the factors are defined as follows: κ is the winding number

$$\kappa = \kappa_\rho = \text{ind}_\rho(F) = \frac{1}{2\pi} [\arg F(\rho e^{i\theta})]_0^{2\pi}. \quad (4.4)$$

$$\tilde{F}(z) = z^{-\kappa} F(z). \quad (4.5)$$

The constant factor is the geometric mean

$$G_\rho(\tilde{F}) = \exp \left[\frac{1}{2\pi} \int_0^{2\pi} \log \tilde{F}(\rho e^{i\theta}) d\theta \right]. \quad (4.6)$$

Finally $F_+(z)$ and its inverse are analytic in $|z| \leq \rho$ and $F_-(z)$ and its inverse are analytic in $|z| \geq \rho$. Moreover, because the factor $G_\rho(\tilde{F})$ is used for normalization, we can take $F_+(0) = F_-(\infty) = 1$. $F_+(z)$ and $F_-(z)$ are supposed to be irreducible. The zeros and poles of $F(z)$ are distributed over the factors $F_+(z)$ and $F_-(z)$ so that zeros and poles in $|z| \geq \rho$ are given to $F_+(z)$ while all the zeros and poles in $|z| \leq \rho$ are given to $F_-(z)$. Note that $\text{ind}_\rho(F_+) = \text{ind}_\rho(F_-) = \text{ind}_\rho(\tilde{F}) = 0$. It was shown in [5] that we have

THEOREM 4.1 For $F(z)$ as described in (4.1-6), there exists a nonzero constant c , not depending on n , such that for $n \rightarrow \infty$ we have

•

$$\det T_n^{(m)} = c [G_\rho(\tilde{F})]^{n+1} \{1 + o(1)\} \quad \text{for } m = \kappa. \quad (4.7)$$

•

$$\det T_n^{(m)} = c [G_\rho(\tilde{F})]^{n+1} [(-\xi_1)(-\xi_2) \cdots (-\xi_{m-\kappa})]^{-n} \cdot \{1 + O(|\xi_{m-\kappa}/\sigma|^n)\}, \quad (4.8)$$

if $0 < m - \kappa < Z^+ + 1$ and $|\xi_{m-\kappa}| < \sigma < |\xi_{m-\kappa+1}|$.

$$\det T_n^{(m)} = c [G_\rho(\tilde{F})]^{n+1} [(-\xi_0)(-\xi_{-1}) \cdots (-\xi_{m-\kappa+1})]^n \cdot \{1 + O(|\sigma/\xi_{m-\kappa+1}|^n)\}, \quad (4.9)$$

if $0 < \kappa - m < Z^- + 2$ and $|\xi_{m-\kappa}| < \sigma < |\xi_{m-\kappa+1}|$.

Proof: See [5].

□

We directly have from this theorem the following corollary:

COROLLARY 4.2 For n sufficiently large we have

$$\det T_n^{(m)} \neq 0 \text{ if } -Z^- - 1 \leq m - \kappa < Z^+ + 1 \text{ and } |\xi_{m-\kappa}| < |\xi_{m-\kappa+1}|.$$

Proof: The correction term in the asymptotic expressions (4.8) and (4.9) will almost vanish for n sufficiently large.

□

From this corollary we conclude that the Schur parameters defined in (3.2) and the series defined in (3.3) exist for n sufficiently large and for m as specified in corollary 4.2.

From the asymptotic expressions of Theorem 4.1 we find the following interesting property.

THEOREM 4.3 For $F(z)$ as defined in (4.1-6) and for m satisfying

$$|\xi_m| < |\xi_{m+1}| < |\xi_{m+2}|, -Z^- \leq m \leq Z^+ - 1$$

we have

$$\lim_{n \rightarrow \infty} \alpha_{n-1}^{(m+\kappa)} / \alpha_n^{(m+\kappa)} = \xi_{m+1}. \quad (4.10)$$

If m satisfies $|\xi_m| < |\xi_{m+1}|$, $-Z^- - 1 \leq m \leq Z^+ - 1$, we have

$$\lim_{n \rightarrow \infty} \alpha_n^{(m+\kappa)} \hat{\alpha}_n^{(m+\kappa)} = 0. \quad (4.11)$$

Proof: The Schur parameters do exist for n sufficiently large by corollary 4.2. If we take (3.2a) and fill in the asymptotic expressions of theorem 4.1 we obtain (4.10).

For (4.11) use again (3.2) and the Jacobi identity for Toeplitz determinants [9]

$$(\det T_n^{(m)})^2 = \det T_n^{(m-1)} \det T_n^{(m+1)} + \det T_{n+1}^{(m)} \det T_{n-1}^{(m)} \quad (4.12)$$

to find

$$\alpha_n^{(m+\kappa)} \hat{\alpha}_n^{(m+\kappa)} = 1 - \frac{\det T_{n+1}^{(m)} \det T_{n-1}^{(m)}}{(\det T_n^{(m)})^2}.$$

Then fill in the asymptotics and (4.11) follows. □

This is however a special case of more general results that will be derived in the next sections. To this end we give here one more result concerning the asymptotic behavior of Toeplitz determinants that will turn out to be useful.

THEOREM 4.4 Let $F(z)$ be as in (4.1-6). Then there exist constants c and λ only depending on F such that for $m - \kappa \geq 0$

$$\det T_n^{(m)}(F) = c \lambda^{n+1} \det T_n^{(m-\kappa)}(F_+) \{1 + o(1)\} \text{ as } n \rightarrow \infty.$$

Proof: Using the techniques of [5] we derive from theorem 3 of [2] that

$$\det T_n^{(m)}(F) = \det T_n^{(m-\kappa)}(\tilde{F}) = c(F) [G_\rho(\tilde{F})]^{n+1} (-1)^{(n+m-\kappa)(m-\kappa)}.$$

$$\{ \det T_{m-1-\kappa}^{(n+1)}(F_+^{-1}) + o(1) \}, \quad n \rightarrow \infty$$

and thus also

$$\det T_n^{(m-\kappa)}(F_+) = c(F_+)[G_\rho(F_+)]^{n+1}(-1)^{(n+m-\kappa)(m-\kappa)}.$$

$$\{\det T_{m-1-\kappa}^{(n+1)}(F_+^{-1}) + o(1)\}, \quad n \rightarrow \infty.$$

Hence the theorem is true if we take $c = c(\tilde{F})/c(F_+)$ and $\lambda = G_\rho(\tilde{F})/G_\rho(F_+)$.

□

5. Padé approximation and Rutishauser polynomials

The more general results referred to in the previous section are convergence results that are very similar to the Rutishauser rules [21] for computing the poles of a meromorphic function. A recent proof for them was given by Seewald [23]. These Rutishauser rules are intimately related with Padé approximation. We shall need some properties from this theory and therefore introduce this concept. Let $F(z) = \sum_0^\infty f_k z^k$ be a formal power series. Then we call $P_n^{(m)}(z)/Q_n^{(m)}(z)$ with $P_n^{(m)}(z)$ a polynomial of degree at most m and $Q_n^{(m)}(z)$ a polynomial of degree at most n an (m/n) Padé approximant for $F(z)$ if its McLaurin series is of the form

$$P_n^{(m)}(z)/Q_n^{(m)}(z) = f_0 + f_1 z + \dots + f_{m+n} z^{m+n} + f'_{m+n+1} z^{m+n+1} + \dots$$

Our notation is deliberately chosen to match the previous notation because it is easily seen that if the formal Laurent series in section 2 has $f_k = 0$ for $k < 0$, then the $P_n^{(m)}(z)$ and $Q_n^{(m)}(z)$ found there are precisely the numerator and denominator of the (m/n) Padé approximant for $F(z)$ if only $m \geq 0$. We shall call a formal power series normal if all the Toeplitz matrices $T_n^{(m)}$ are nonsingular for $m, n = 0, 1, 2, \dots$. For a normal power series Padé approximants exist for all $m, n = 0, 1, 2, \dots$. Note that numerator and denominator of a Padé approximant can always be multiplied with a nonzero constant. The normalization used in section 2 is that $Q_n^{(m)}(z)$ is comonic (with constant term equal to 1). From their definitions it is clear that $\hat{Q}_n^{(m+1)}(z)$ and $\hat{P}_n^{(m+1)}(z)$ satisfy the same

relations as $Q_n^{(m)}(z)$ and $P_n^{(m)}(z)$ but now with a monic normalization for the denominator. Indeed,

$$S_n^{(m)}(z) = \hat{S}_n^{(m+1)}(z)/(-\hat{\alpha}_n^{(m+1)}) = -\alpha_n^{(m)}\hat{S}_n^{(m+1)} \quad (5.1)$$

for S one of P, Q or R . Introduce (5.1) into (2.2a) and we get

$$\hat{S}_{n+1}^{(m)}(z) = z\hat{S}_n^{(m)}(z) - \hat{b}_{n+1}^{(m)}\hat{S}_n^{(m-1)}(z) \quad (5.2)$$

$$\text{with } \hat{b}_{n+1}^{(m)} = -\hat{\alpha}_{n+1}^{(m)}\alpha_n^{(m)} \quad (5.3)$$

For a normal power series, all these numbers $\hat{b}_n^{(m)}$ do exist and they are used to define the Rutishauser polynomials by the recurrence relation

$$\hat{Q}_{n,n+k+1}^{(m)}(z) = z\hat{Q}_{n,n+k}^{(m)}(z) - \hat{b}_{n+k+1}^{(m)}\hat{Q}_{n,n+k}^{(m+1)}(z), k \geq 1 \quad (5.4)$$

with initial conditions $\hat{Q}_{n,n}^{(m)} = 1$.

It is obvious that $\hat{Q}_{n,n+k}^{(m)}(z)$ is a polynomial of degree k and as a special case we get $\hat{Q}_{0,k}^{(m)}(z) = \hat{Q}_k^{(m)}(z)$. A simple proof by induction can show that we have, at least in the normal case, that $\hat{S}_n^{(m)}(z)$ with $S = P, Q$ or R satisfies the relation

$$\hat{S}_{n+h}^{(m)}(z) = \sum_{i=0}^h \hat{q}_i z^i \hat{S}_n^{(m+h-i)}(z) \quad (5.5)$$

if

$$\hat{Q}_{n,n+h}^{(m)}(z) = \sum_{i=0}^h \hat{q}_i z^i \quad (5.6)$$

is a Rutishauser polynomial as defined in (5.4).

It is a shortcoming of this treatment of Rutishauser polynomials that they are only defined for a normal power series. However if the polynomials $\hat{Q}_{n+h}^{(m)}(z)$ and $\hat{Q}_n^{(m-i)}(z)$, $i = 0, 1, \dots, h$ are well defined, then we shall consider (5.5-6) as a definition of $\hat{Q}_{n,n+h}^{(m)}(z)$. It is a well known fact [13] that if $F(z)$ is as in (4.1-6) with $F_-(z) = 1$, $\kappa = 0$ and $|\pi_n| < |\pi_{n+1}|$,

then $\det T_n^{(m)} \neq 0$ for m sufficiently large (this is actually a key element in the proof of theorem 4.1). Hence $\hat{Q}_n^{(m)}(z)$ then exists by (3.3a). This implies that if not only $|\pi_n| < |\pi_{n+1}|$, but also $|\pi_{n+h}| < |\pi_{n+h+1}|$, the Rutishauser polynomials $\hat{Q}_{n,n+h}^{(m)}(z)$ exist for m sufficiently large. For a normal power series, the coefficients of $\hat{Q}_{n,n+h}^{(m)}(z)$ are made up of sums and products of some of the $\hat{b}_n^{(m)}$. The previous statement says that for a nonnormal power series, even if some of these $\hat{b}_n^{(m)}$ do not exist, their combinations forming the coefficients of $\hat{Q}_{n,n+h}^{(m)}(z)$ will exist for m sufficiently large.

The original Rutishauser approach worked with the polynomials $Q_n^{(m)}(z)$ instead of $\hat{Q}_n^{(m)}(z)$ as we did. The definition of those Rutishauser polynomials is slightly different. Our normalization is more like Gragg's [9]. The convergence result proved by Seewald [24] for the original Rutishauser polynomials reads for our normalization as follows:

THEOREM 5.1 Let $F(z)$ be as in (4.1-6) with $F_-(z) = 1$ and $\kappa = 0$. Define for $0 \leq n \leq n+h \leq P^+$, $|\pi_n| < |\pi_{n+1}|$ and $|\pi_{n+h}| < |\pi_{n+h+1}|$ the polynomial

$$\hat{Q}_{n,n+h}(z) = (z - \pi_{n+1})(z - \pi_{n+2}) \cdots (z - \pi_{n+h}). \quad (5.7)$$

Then the Rutishauser polynomial $\hat{Q}_{n,n+h}^{(m)}(z)$ defined by (5.5-6) exists for m sufficiently large and

$$|\hat{Q}_{n,n+h}^{(m)}(z) - \hat{Q}_{n,n+h}(z)| \leq C \lambda^m \quad \forall z \in K, \quad m \text{ large}, \quad (5.8)$$

where C is a positive constant, K a compact subset of $D = \{z: 0 < |z| < \infty\}$ and λ arbitrary but

$$1 > \lambda > \max(|\pi_n/\pi_{n+1}|, |\pi_{n+h}/\pi_{n+h+1}|). \quad (5.9)$$

Proof: See [24].

□

As an example of this theorem, suppose that $h = 1$, hence

$\hat{Q}_{n,n+1}(z) = z - \pi_{n+1}$. Now $\hat{Q}_{n,n+1}^{(m)}(z) = z - \hat{b}_{n+1}^{(m)} = z - \hat{\alpha}_{n+1}^{(m)}\alpha_n^{(m)}$ and because of the uniform convergence, it follows that

$$\lim_{m \rightarrow \infty} \hat{\alpha}_{n+1}^{(m)}\alpha_n^{(m)} = -\pi_{n+1}. \quad (5.10)$$

This is precisely the type of result we are after but we would like to have it for the zeros of $F(z)$, instead of the poles and we want n to go to infinity instead of m . The next section solves this problem for a power series.

6. The $\pi\zeta$ algorithm and Rutishauser rules for zeros

A method to obtain the zeros of $F(z)$ from Rutishauser polynomials would be by inversion of $F(z)$ and finding the zeros of $F(z)$ as the poles of $1/F(z)$. This would introduce Schur parameters for $1/F(z)$ and it is not clear at the first sight how these are related to the Schur parameters of $F(z)$. There is however a simple symmetry relation between $F(z)$ and $1/F(z)$ for two other sets of parameters that can be expressed in terms of the Schur parameters for these series. One of these sets contains the numbers $\hat{b}_n^{(m)}$ introduced in the previous section by (5.3). The other set contains numbers $\hat{a}_n^{(m)}$. These appear in a recurrence relation that can also be derived from (2.2a) with (5.1) just like (5.2) was found. It is

$$\hat{S}_{n+1}^{(m+1)}(z) = \hat{S}_{n+1}^{(m)}(z) + \hat{a}_{n+1}^{(m)}\hat{S}_n^{(m+1)}(z) \quad (6.1)$$

with

$$\hat{a}_{n+1}^{(m)} = (1 - \alpha_n^{(m)}\hat{\alpha}_n^{(m)})\alpha_{n-1}^{(m)}/\alpha_n^{(m)}. \quad (6.2)$$

and S one of P, Q or R . Again all the numbers $\hat{a}_n^{(m)}$ exist for $n \geq 1$ and $m \geq 0$ if we set by definition $\alpha_0^{(m)} = -1$ and if $F(z)$ is a normal power series. These numbers satisfy some rhombus rules. These rules, together with initial conditions form the basis for the $\pi\zeta$ algorithm introduced by Gragg [9].

THEOREM 6.1 If $F(z)$ is a normal Laurent series, then the numbers $\hat{a}_n^{(m)}$ and $\hat{b}_n^{(m)}$ defined in (5.3) and (6.2) satisfy the rhombus rules

$$\hat{a}_k^{(m)} \hat{a}_{k-1}^{(m+1)} = \hat{a}_k^{(m+1)} \hat{b}_k^{(m+1)}, \quad k \geq 2 \quad (6.3a)$$

$$\hat{a}_{k+1}^{(m)} + \hat{b}_{k+1}^{(m+1)} = \hat{a}_k^{(m)} + \hat{b}_k^{(m)}, \quad k \geq 1 \quad (6.3b)$$

with initial conditions

$$\hat{b}_1^{(m+1)} = f_m / f_{m+1} ; \quad \hat{a}_1^{(m)} = 0. \quad (6.4)$$

Proof: If you plug the determinant formulas for $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$ from (3.2) into the definitions (5.2) and (6.2) you get

$$\hat{a}_{n+1}^{(m)} = - \frac{\det T_{n-2}^{(m+1)} \det T_n^{(m)}}{\det T_{n-1}^{(m+1)} \det T_{n-1}^{(m)}} \quad (6.5a)$$

and

$$\hat{b}_{n+1}^{(m)} = \frac{\det T_n^{(m-1)} \det T_{n-1}^{(m+1)}}{\det T_n^{(m)} \det T_{n-1}^{(m)}}. \quad (6.5b)$$

Use can be made of the identity (4.12). With the expressions (6.5) and some algebra, (6.3) and (6.4) are easily obtained. □

If $F(z)$ is a power series instead of a Laurent series, the same relations hold for the $\hat{a}_n^{(m)}$ and $\hat{b}_n^{(m)}$ numbers as far as they make sense. That is for $m \geq 0$ and $n \geq 1$. There is a striking symmetry between these numbers defined for a power series $F(z)$ with $f_0 \neq 0$ and its formal inverse $G(z) = 1/F(z)$. To distinguish between both, we add the argument F or G as appropriate.

THEOREM 6.2 For $F(z) = \sum_0^\infty f_k z^k$ a normal power series with $f_0 \neq 0$ and $G(z) = \sum_0^\infty g_k z^k$ its formal inverse, define the numbers $\hat{a}_{n+1}^{(m)}(F)$ and

$\hat{b}_{n+1}^{(m)}(F)$ for $m, n \geq 0$ by (5.2) and (6.2) and $\hat{a}_{n+1}^{(m)}(G)$ and $\hat{b}_{n+1}^{(m)}(G)$ are similarly defined for $G(z)$. Then

$$\hat{a}_{n+1}^{(m)}(F) = \hat{b}_{m+1}^{(n)}(G) \text{ for } m, n \geq 0. \quad (6.6)$$

Proof: It follows from (2.1) that $\hat{Q}_n^{(0)}(z; F) = z^n$, so that

$$\hat{Q}_n^{(0)}(0; F) = -\hat{\alpha}_n^{(0)}(F) = 0 \text{ for } n \geq 1 \text{ and } \hat{\alpha}_0^{(0)}(F) = -1. \quad (6.7)$$

From (2.12a) we get after multiplication with $G(z)$:

$$Q_n^{(0)}(z; F) - G(z)P_n^{(0)}(z; F) = G(z)R_n^{(0)}(z; F).$$

Since the right hand side contains only powers in z^k for $k > n$, and because $P_n^{(0)}(z; F) = f_0$, we get by equating the coefficients of z^n that

$$-\alpha_n^{(0)}(F) - g_n f_0 = 0 \text{ or } \alpha_n^{(0)}(F) = -f_0 g_n, n \geq 0. \quad (6.8)$$

(6.4), (6.7) and (6.8) give

$$\hat{b}_n^{(0)}(F) = -\hat{\alpha}_n^{(0)}(F)\alpha_{n-1}^{(0)}(F) = 0 = \hat{a}_1^{(n-1)}(G), n \geq 1. \quad (6.9)$$

From (6.1):

$$\hat{Q}_n^{(1)}(z; F) = \hat{Q}_n^{(0)}(z; F) + \hat{a}_{n+1}^{(0)}(F)\hat{Q}_{n-1}^{(1)}(z; F).$$

The constant term in this expression gives for $n \geq 1$

$$\begin{aligned} \hat{a}_{n+1}^{(0)}(F) &= \hat{\alpha}_n^{(1)}(F)/\hat{\alpha}_{n+1}^{(1)}(F) = \alpha_{n-1}^{(0)}(F)/\alpha_n^{(0)}(F) = g_{n-1}/g_n \\ &= \hat{b}_1^{(n)}(G). \end{aligned} \quad (6.10)$$

(6.9) and (6.10) prove (6.6) for the initial conditions. That (6.6) is also true for the other values of m and n follows by induction from the rules (6.3).

□

From the symmetry proved in theorem 6.2 we easily derive some other symmetry relations.

COROLLARY 6.3 Under the same conditions as theorem 6.2, we have the following symmetry relations for numerator and denominator pairs of Padé approximants as defined in (2.1) and (2.10):

$$\hat{Q}_{m-1}^{(n+1)}(z;G) = \hat{P}_n^{(m)}(z;F) := \hat{P}_n^{(m)}(z;F)/\hat{u}_n^{(m)}(F) \quad (6.11a)$$

and

$$\hat{P}_{m-1}^{(n+1)}(z;G) = \hat{Q}_n^{(m)}(z;F) := \hat{Q}_n^{(m)}(z;F)/\hat{u}_n^{(m)}(F), \quad (6.11b)$$

with $\hat{u}_n^{(m)}(F)$ as in (2.4).

Proof: This is a consequence of the simple fact that if $\hat{P}_n^{(m+1)}(z)/\hat{Q}_n^{(m+1)}(z)$ is an (m/n) Padé approximant for $F(z)$, then $\hat{Q}_n^{(m+1)}(z)/\hat{P}_n^{(m+1)}(z)$ is an (n/m) Padé approximant for $G(z)$. This follows from (2.12b) after multiplication with $G(z)$. The factor $1/\hat{u}_n^{(m)}$ is introduced to match the proper normalization.

□

The reason why we derived all these symmetry relations is that we can now translate the results of section 5 into corresponding results for the zeros of $F(z)$ without computing $1/F(z)$. E.g. if the Rutishauser polynomials are defined by (5.4) for $G(z)$, then it follows from theorem 6.2 that exactly the same polynomials are generated by

$$\hat{P}_m^{(n,n+k+1)}(z) = z\hat{P}_m^{(n,n+k)}(z) - \hat{a}_{m+1}^{(n+k)}(F)\hat{P}_{m+1}^{(n,n+k)}(z) \quad (6.12a)$$

with

$$\hat{P}_m^{(n,n)} = 1 \quad (6.12b)$$

and we can identify $\hat{P}_m^{(0,k)}(z)$ with $\hat{P}_{m-1}^{(k+1)}(z) := \hat{P}_{m-1}^{(k+1)}(z)/\hat{u}_{m-1}^{(k+1)}$. The analog of (5.5-6) becomes for S one of P, Q or R

$$\hat{S}_{m-1}^{(n+h+1)}(z) = \sum_{i=0}^h \hat{q}_i z^i \hat{S}_{m+h-i-1}^{(n+1)}(z), \quad (6.13)$$

with

$$\hat{S}_n^{(m)}(z) := \hat{S}_n^{(m)}(z) / \hat{u}_n^{(m)}$$

and

$$\hat{P}_m^{(n, n+h)}(z) = \sum_{i=0}^h \hat{q}_i z^i, \quad (6.14)$$

a Rutishauser polynomial defined by (6.12). Note that $\hat{S}_n^{(m)}(z)$ exists if $\det T_n^{(m-1)} \neq 0$. Again we may consider (6.13) as a definition for $\hat{P}_m^{(n, n+h)}(z)$ if $\hat{S}_{m-1}^{(n+h+1)}(z)$ and $\hat{S}_{m-i-1}^{(n+1)}(z)$, $i = 0, 1, \dots, h$ exist. Skipping further details, we give the symmetric version of theorem 5.1 without proof.

THEOREM 6.4 Let $F(z)$ be as in (4.1-6) with $F_-(z) = 1$ and $\kappa = 0$. Define for $0 < m \leq m+h \leq Z^+$, $|\zeta_m| < |\zeta_{m+1}|$ and $|\zeta_{m+h}| < |\zeta_{m+h+1}|$ the polynomial

$$\hat{P}^{(m, m+h)}(z) = (z - \zeta_{m+1})(z - \zeta_{m+2}) \cdots (z - \zeta_{m+h}). \quad (6.15)$$

Then the Rutishauser polynomial $\hat{P}_n^{(m, m+h)}(z)$ defined by (6.13-14)) exists for n sufficiently large and

$$|\hat{P}_n^{(m, m+h)}(z) - \hat{P}^{(m, m+h)}(z)| \leq C \lambda^n \quad \forall z \in K, \quad n \text{ large}, \quad (6.16)$$

where C is a positive constant, K a compact subset of $D = \{z: 0 < |z| < \infty\}$ and λ arbitrary but

$$1 > \lambda > \max(|\zeta_m / \zeta_{m+1}|, |\zeta_{m+h} / \zeta_{m+h+1}|). \quad (6.17)$$

□

(4.10) follows from this just like (5.10) followed from theorem 5.1, at least if $F(z)$ is a power series.

7. Computing zeros from a Laurent series

In the two preceding sections we were concerned with power series of a meromorphic function, while in our original problem formulation we had a Laurent series. So we are left with the problem of extending the results of last section to the case of a Laurent series. Here we use theorem 4.4 relating a Toeplitz determinant based on a Laurent series for $F(z)$ to a Toeplitz determinant based on a Laurent series for $F_+(z)$ which is simply a power series. A consequence of this theorem is the following.

THEOREM 7.1 Let $F(z)$ be as defined in (4.1-6) and suppose $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$ are defined by (3.2) and $\hat{a}_n^{(m)}$ and $\hat{b}_n^{(m)}$ are defined by (6.5) where the Toeplitz determinants are based on either $F(z)$ or $F_+(z)$. Then for $0 \leq m - \kappa \leq Z^+ + 1$ with an obvious notation

$$s_{n+1}^{(m)}(F) = s_{n+1}^{(m-\kappa)}(F_+) \{1+o(1)\} \text{ for } n \rightarrow \infty, \quad (7.1)$$

where s is one of $\alpha, \hat{\alpha}, \hat{b}$ or $\hat{a}$.

Proof: It is sufficient to give the proof for $s = \alpha$ because the others are completely similar. From (3.2a) we have for n sufficiently large and for m within the bounds indicated :

$$\alpha_{n+1}^{(m)}(F) = (-1)^n \det T_n^{(m+1)}(F) / \det T_n^{(m)}(F).$$

By theorem 4.4 we may replace $\det T_n^{(m+1)}(F)$ by $C \lambda^{n+1} \det T_n^{(m+1-\kappa)}(F_+) \{1+o(1)\}$ and similarly for $\det T_n^{(m)}(F)$ if $n \rightarrow \infty$. Thus

$$\begin{aligned} \alpha_{n+1}^{(m)}(F) &= (-1)^n \det T_n^{(m+1-\kappa)}(F_+) / \det T_n^{(m-\kappa)}(F_+) \{1+o(1)\} \\ &= \alpha_{n+1}^{(m-\kappa)}(F_+) \{1+o(1)\}. \end{aligned}$$

□

This gives us directly the following corollary:

COROLLARY 7.2 For the Rutishauser polynomials defined in (6.13-14) where we suppose that the same definitions are applied if $F(z)$ is a Laurent series, we have for $n \rightarrow \infty$ and $0 \leq m - \kappa \leq m - \kappa + k \leq Z^+$

$$\hat{P}_n^{(m, m+k)}(z; F) = \hat{P}_n^{(m-\kappa, m-\kappa+k)}(z; F_+) \{1 + o(1)\}. \quad (7.2)$$

$F(z)$ is a general meromorphic function as described in (4.1-6).

Proof: Recall the remark we made when defining the Rutishauser polynomial $\hat{Q}_{n, n+h}^{(m)}(z)$ in section 5. Whenever this polynomial exists, its coefficients are made up of combinations of $\hat{b}_n^{(m)}$ numbers that may not all exist individually but the combinations do exist for m sufficiently large. We can similarly conclude from (6.13-14) that the Rutishauser polynomial $\hat{P}_n^{(m, m+h)}(z)$ will have coefficients that are made up of combinations of $\hat{a}_n^{(m)}$ numbers. Some of these may be infinite but the combinations will be such that they exist. The previous theorem stated that each $\hat{a}_n^{(m)}(F)$ differs only by an $o(1)$ correction from $\hat{a}_n^{(m-\kappa)}(F_+)$ for $n \rightarrow \infty$. Hence the coefficients of the polynomials of (7.2) will only differ by terms that are $o(1)$ for $n \rightarrow \infty$. □

We can now generalize the results of previous section to the Laurent case.

THEOREM 7.3a Let $F(z)$ be as in (4.1-6). Define for $|\zeta_{m-\kappa}| < |\zeta_{m-\kappa+1}|$ and $|\zeta_{m-\kappa+h}| < |\zeta_{m-\kappa+h+1}|$ where $0 \leq m - \kappa < m - \kappa + h \leq Z^+$ the polynomial

$$\hat{P}^{(m, m+h)}(z) = (z - \zeta_{m-\kappa+1})(z - \zeta_{m-\kappa+2}) \cdots (z - \zeta_{m-\kappa+h}). \quad (7.3)$$

Let $\hat{P}_n^{(m, m+k)}(z)$ be a Rutishauser polynomial for $F(z)$ defined by (6.13-14). Then

$$|\hat{P}_n^{(m, m+h)}(z) - \hat{P}^{(m, m+h)}(z)| \leq C \lambda^n, \quad \forall z \in K, n \text{ large}, \quad (7.4)$$

where C is a positive constant, K a compact subset of $\{z: 0 < |z| < \infty\}$ and

$$1 > \lambda > \max (|\zeta_{m-\kappa}/\zeta_{m-\kappa+1}|, |\zeta_{m-\kappa+h}/\zeta_{m-\kappa+h+1}|). \quad (7.5)$$

Proof: This is simply a transcription of theorem 6.4 where we took corollary 7.2 into account. $\square$

A completely symmetric development for a power series in $1/z$ can be made. This gives the possibility to find the zeros of $F(z)$ that are in $|z| < \rho$. The Rutishauser polynomials have then to be defined by

$$\begin{aligned} \hat{P}_n^{(m, m-k-1)}(z) &= z \hat{P}_n^{(m, m-k)}(z) - \hat{a}_{n+1}^{(m-k-1)} \hat{P}_{n-1}^{(m, m-k)}(z); \\ \hat{P}_n^{(m, m)} &= 1, \end{aligned} \quad (7.6)$$

where k, m and n are nonnegative integers. Again suitable adaptations of this recursive definition are in order if some of the $\hat{a}_n^{(m)}$ numbers are infinite because $F(z)$ need not be normal. For these polynomials a result quite similar to the one of theorem 7.3a can be obtained. We give it below without further proof.

THEOREM 7.3b Let $F(z)$ be as in (4.1-6). For $|\zeta_{m-\kappa}| < |\zeta_{m-\kappa+1}|$ and $|\zeta_{m-\kappa-h}| < |\zeta_{m-\kappa-h+1}|$ where $0 \geq m-\kappa > m-\kappa-h \geq -Z-1$ define the polynomial

$$\hat{P}^{(m, m-h)}(z) = (z - \zeta_{m-\kappa})(z - \zeta_{m-\kappa-1}) \cdots (z - \zeta_{m-\kappa-h+1}). \quad (7.7)$$

Let $\hat{P}_n^{(m, m+k)}(z)$ be defined by (7.6) or a suitable adaptation. Then

$$|\hat{P}_n^{(m, m-h)}(z) - \hat{P}^{(m, m-h)}(z)| \leq C \lambda^n, \quad \forall z \in K, n \text{ large}, \quad (7.8)$$

where C is a positive constant, K a compact subset of $\{z: 0 < |z| < \infty\}$ and

$$1 > \lambda > \max (|\zeta_{m-\kappa}/\zeta_{m-\kappa+1}|, |\zeta_{m-\kappa-h}/\zeta_{m-\kappa-h+1}|). \quad (7.9)$$

$\square$

8. Convergence results for Schur parameters

The convergence results for the Rutishauser polynomials that were obtained in the last section give us really results on the convergence of the Schur parameters. Indeed, the coefficients of the Rutishauser polynomials are made up of $\hat{a}_n^{(m)}$ numbers, hence of Schur parameters. The result (4.10) is just a special case of more general results that can now be obtained. We give some examples of this in the next theorem

THEOREM 8.1 Let $F(z)$ be as defined in (4.1-6) and the numbers $\hat{a}_n^{(m)}$, $\hat{b}_n^{(m)}$ and the Schur parameters $\alpha_n^{(m)}$ and $\hat{\alpha}_n^{(m)}$ by (6.5) and (3.2). Then, if we denote the limit for $n \rightarrow \infty$ as $\lim$ for short, we have:

1. If $|\zeta_m| < |\zeta_{m+1}|$, $-Z^- - 1 \leq m \leq Z^+$, then

$$\lim \hat{b}_n^{(m+\kappa)} = \lim \alpha_n^{(m+\kappa)} \hat{\alpha}_n^{(m+\kappa)} = 0. \quad (8.1)$$

2. If $|\zeta_m| < |\zeta_{m+1}| < |\zeta_{m+2}|$, $-Z^- - 1 \leq m \leq Z^+ - 1$, then

$$\begin{aligned} \lim \hat{a}_n^{(m+\kappa)} &= \lim \alpha_{n-1}^{(m+\kappa)} / \alpha_n^{(m+\kappa)} \\ &= \lim \hat{\alpha}_n^{(m+\kappa+1)} / \hat{\alpha}_{n-1}^{(m+\kappa+1)} = \zeta_{m+1}. \end{aligned} \quad (8.2)$$

3. If $|\zeta_m| < |\zeta_{m+1}| \leq |\zeta_{m+2}| < |\zeta_{m+3}|$, $-Z^- - 1 \leq m \leq Z^+ - 2$, then

$$\lim (\hat{a}_n^{(m+\kappa)} + \hat{a}_n^{(m+\kappa+1)}) = \zeta_{m+1} + \zeta_{m+2}, \quad (8.3a)$$

$$\lim \hat{a}_{n+1}^{(m+\kappa)} \hat{a}_n^{(m+\kappa+1)} = \zeta_{m+1} \zeta_{m+2}. \quad (8.3b)$$

Proof: (8.1) was already proved as (4.11) if you use the relation (5.3). (8.2) was partially shown as (4.10). We can also find it from theorem 7.3. Indeed, suppose $m > \kappa$, then we find from theorem 7.3a with $h = 1$ that $\hat{P}_n^{(m, m+1)}(z) = z - \hat{a}_{n+1}^{(m)}$ converges to $z - \zeta_{m+1}$ for $n \rightarrow \infty$. Hence $\lim \hat{a}_{n+1}^{(m)} = \zeta_{m+1}$. (6.2) and the first part of this theorem prove (8.2) completely. For $m \leq \kappa$ a similar proof can be found. To prove (8.3a,b)

use the same technique but now with $h = 2$.

□

This theorem will make clear how many other results on the convergence of the Schur parameters could be found.

We remark here that the convergence results for the Rutishauser polynomials as they were obtained in theorems 7.3 can also be used to find under certain conditions the limits of the polynomials $Q_n^{(m)}(z)$ and $\hat{Q}_n^{(m)}(z)$ for n going to infinity. The norm convergence of these was treated in [8] where it was shown that we can only expect convergence for one particular m . The convergence that can be obtained from our results is valid for more m values but it is a uniform convergence in certain compact sets of the complex plane. The derivation of these results at this place would lead us too far. They will be published elsewhere.

9. References

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