ON THE BLOCK STRUCTURE OF THE LAURENT-PADE TABLE

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Abstract. Some ideas on the block structure of a formal Laurent-Padé table will be given. The structure is derived from the block structure of the table of Toeplitz determinants which also defines the blocks of a classical Padé table.

1. Formal Laurent series

With a bi-infinite sequence of complex numbers $\{f_k\}_{k=-\infty}^{\infty}$ we associate a formal *Laurent series* (fls)

$$F(z) = \sum_{k=-\infty}^{\infty} f_k z^k. \quad (1.1)$$

$F(z)$ will be fixed throughout this paper. With F we associate the series

$$Z(z) = \frac{1}{2} f_0 + \sum_{k=1}^{\infty} f_k z^k \quad \text{and} \quad \hat{Z}(z) = \frac{1}{2} f_0 + \sum_{k=1}^{\infty} f_{-k} z^{-k}. \quad (1.2)$$

A *projection operator* $\Pi_{m:n}$ is defined by

$$\Pi_{m:n} F(z) = \sum_{k=m}^n f_k z^k. \quad (1.3)$$

A *Laurent polynomial* is an expression like in the right hand side of (1.3) with m and n finite. Its *degree* is $\max \{|c| \mid f_c \neq 0\}$. The notations O_+ and O_- are introduced via

$$\begin{aligned} F(z) = O_+ (z^m) \quad & \text{iff} \quad \Pi_{-\infty:m-1} F(z) = 0, \\ F(z) = O_- (z^n) \quad & \text{iff} \quad \Pi_{n+1:\infty} F(z) = 0. \end{aligned} \quad (1.4)$$

$\text{Sup } \{m | F(z) = O_+(z^m)\}$ is called the *+ order* of $F(z)$ and we denote it as $\text{ord}_+(F)$. Similarly $\text{Inf } \{n | F(z) = O_-(z^n)\}$ is called the *- order* of $F(z)$ and we denote it as $\text{ord}_-(F)$.

The *product* of two fls is defined in the usual way as a Cauchy product, for those cases where both have finite + order, finite - order or if one of them is a Laurent polynomial. This is all that we shall need. The *reciprocal* of a fls with finite + order (- order) is also defined in the usual way. It is again a fls with finite + order (- order). If it is a Laurent polynomial, it is not clear which reciprocal we want : the one with finite + order or the one with finite - order. To distinguish between both we use the notation $L_+(1/F(z))$ to indicate the reciprocal with finite + order and $L_-(1/F(z))$ for the other one. Also if $F(z)$ and $G(z)$ are Laurent polynomials, then

$$L_+(F(z)/G(z)) \text{ means } F(z) L_+(1/G(z)) \text{ etc...}$$

By a *ratio of two Laurent polynomials* we mean a complete equivalence class. I.e. we do not distinguish between $P(z)/Q(z)$ and $[A(z)P(z)]/[A(z)Q(z)]$ for $A(z) \neq 0$ and $P_1(z)/Q_1(z) = P_2(z)/Q_2(z)$ means that they are in the same equivalence class, i.e. $P_1(z)Q_2(z) = P_2(z)Q_1(z)$.

We call the fls (1.1) normal if all the Toeplitz determinants $T_{mn} = \det(f_{m+i-j})_{i,j=1}^n$ for $m = 0, \pm 1, \pm 2, \dots; n = 1, 2, \dots$ are nonzero. It will be convenient to define $T_{m0} = 1$.

2. Laurent-Padé Approximants and Laurent-Padé Forms

The notion of Padé approximant for a formal power series can be extended to what is called a Laurent-Padé approximant (LPA) for a fls. Let m and n be nonnegative integers and suppose $R(z)$ and $\hat{R}(z)$ are both ratios of two Laurent polynomials such that :

$$\begin{aligned} (1) \quad Z(z) - L_+(R(z)) &= O_+(z^{m+n+1}), \\ (2) \quad \hat{Z}(z) - L_-(\hat{R}(z)) &= O_-(z^{-(m+n+1)}), \\ (3) \quad R(z) + \hat{R}(z) &= A_m(z)/B_n(z) \text{ where } A_m(z) \text{ and } B_n(z) \text{ are Laurent polynomials of} \\ &\text{degree at most } m, \text{ respectively } n. \end{aligned} \tag{2.1}$$

Then we call the pair $(R(z), \hat{R}(z))$ an (m, n) LPA for $F(z)$.

Remark 1 : This is clearly an extension of the classical notion of Padé approximant since $L_+(R(z)) + L_-(\hat{R}(z))$ is a fls which matches the coefficients f_k for $|k| \leq m+n$ in the given fls while $R(z) + \hat{R}(z)$ is the ratio of two Laurent polynomials with numerator degree m and denominator degree n .

Remark 2 : We preferred to define a LPA as a couple of ratios rather than as the sum of these because in our formal setting there is not an unambiguous way to return from $A_m(z)/B_n(z)$ to $R(z)$ and $\hat{R}(z)$, thus to the fls that we associate with it. A LPA may not exist. What always exists is a Laurent-Padé form (LPF) which we now define by relaxing the conditions (1) and (2).

We call $(R(z), \hat{R}(z))$ an (m, n) LPF if $R(z)$ and $\hat{R}(z)$ have representants

$R(z) = P_{mn}(z)/Q_{mn}(z)$ and $\hat{R}(z) = \hat{P}_{-m,n}(z)/\hat{Q}_{-m,n}(z)$, such that

$$(1) \quad Z(z) Q_{mn}(z) - P_{mn}(z) = O_+(z^{m+n+1}),$$

$$(2) \quad \hat{Z}(z) \hat{Q}_{-m,n}(z) - \hat{P}_{-m,n}(z) = O_-(z^{-(m+n+1)}),$$

(3) is as above.

A constructive way to find the (m, n) LPF goes as follows :

Theorem 2.1 Let $\mu = 0, \pm 1, \pm 2, \dots$; $\nu = 0, 1, 2, \dots$

Let $Q_{\mu\nu}(z)$ and $\hat{Q}_{\mu\nu}(z)$ be nontrivial polynomials of degree ν which are (not necessarily unique) solutions of

$$\Pi_{\mu+1:\mu+\nu} (F(z) Q_{\mu\nu}(z)) = 0 \quad \text{and} \quad \Pi_{\mu:\mu+\nu-1} (F(z) \hat{Q}_{\mu\nu}(z)) = 0. \quad (2.2)$$

Define Laurent polynomials $P_{\mu\nu}(z)$ and $\hat{P}_{\mu\nu}(z)$ by

$$P_{\mu\nu}(z) = \Pi_{k(\mu,\nu):\ell(\mu,\nu)} (Z(z) Q_{\mu\nu}(z)) \quad (2.3a)$$

and

$$\hat{P}_{\mu+1,\nu}(z) = \Pi_{k(\mu,\nu):\ell(\mu,\nu)} (\hat{Z}(z) \hat{Q}_{\mu+1,\nu}(z)), \quad (2.3b)$$

where $k(\mu, \nu) = \min(0, \mu+\nu+1)$ and $\ell(\mu, \nu) = \max(\mu, \nu)$.

Then $K_{mn}(z) = (P_{mn}(z)/Q_{mn}(z), \hat{P}_{-m,n}(z)/\hat{Q}_{-m,n}(z))$ and

$\hat{K}_{m+1,n}(z) = (\hat{P}_{m+1,n}(z)/\hat{Q}_{m+1,n}(z), P_{-1-m,n}(z)/Q_{-1-m,n}(z))$ are (m, n) LPF's for $m \geq 0$.

Proof

We remark that equations (2.1) always have nontrivial solutions. We define auxiliary formal series by

$$V_{\mu\nu}(z) = \Pi_{-\infty:\mu} (F(z) Q_{\mu\nu}(z)), \quad \hat{V}_{\mu\nu}(z) = \Pi_{-\infty:\mu-1} (F(z) \hat{Q}_{\mu\nu}(z)), \quad (2.4)$$

$$W_{\mu\nu}(z) = \Pi_{\mu+\nu+1:\infty} F(z) Q_{\mu\nu}(z), \quad \hat{W}_{\mu\nu}(z) = \Pi_{\mu+\nu:\infty} (F(z) \hat{Q}_{\mu\nu}(z)). \quad (2.5)$$

Clearly $F(z)Q_{\mu\nu}(z) = V_{\mu\nu}(z) + W_{\mu\nu}(z)$ and $F(z)\hat{Q}_{\mu\nu}(z) = \hat{V}_{\mu\nu}(z) + \hat{W}_{\mu\nu}(z)$.

From this it follows that (remember $F(z) = Z(z) + \hat{Z}(z)$)

$$Z(z)Q_{\mu\nu}(z) - W_{\mu\nu}(z) = -\hat{Z}(z)Q_{\mu\nu}(z) + V_{\mu\nu}(z) = P_{\mu\nu}(z), \quad (2.6)$$

$$Z(z)\hat{Q}_{\mu\nu}(z) - \hat{W}_{\mu\nu}(z) = -\hat{Z}(z)\hat{Q}_{\mu\nu}(z) + \hat{V}_{\mu\nu}(z) = -\hat{P}_{\mu\nu}(z). \quad (2.7)$$

These could be seen as alternatives for the definitions (2.3).

From this we find that

$$Z(z)Q_{mn}(z) - P_{mn}(z) = W_{mn}(z) = O_+(z^{m+n+1})$$

and

$$\hat{Z}(z)(z^{-n}\hat{Q}_{-m,n}(z)) - (z^{-n}\hat{P}_{-m,n}(z)) = z^{-n}\hat{V}_{-m,n}(z) = O_-(z^{-(m+n+1)}),$$

so that the order of approximation is as required. Consider now

$$\frac{P_{mn}(z)}{Q_{mn}(z)} + \frac{z^{-n}\hat{P}_{-m,n}(z)}{z^{-n}\hat{Q}_{-m,n}(z)} = \frac{z^{-n}(P_{mn}(z)\hat{Q}_{-m,n}(z) + \hat{P}_{-m,n}(z)Q_{mn}(z))}{z^{-n}Q_{mn}(z)\hat{Q}_{-m,n}(z)}.$$

The degree of the Laurent polynomial in the denominator is as it should be. Let L_{mn} be the numerator. Then we only have to prove that it is a Laurent polynomial of degree at most m . We drop the argument in the notation for simplicity. For $m, n \geq 0$ we have

$$\begin{aligned} z^n L_{mn} &= [Z Q_{mn} - W_{mn}] \hat{Q}_{-m,n} + [\hat{Z} \hat{Q}_{-m,n} - \hat{V}_{-m,n}] Q_{mn} \\ &= F Q_{mn} \hat{Q}_{-m,n} - W_{mn} \hat{Q}_{-m,n} - \hat{V}_{-m,n} Q_{mn} \\ &= [V_{mn} + W_{mn}] \hat{Q}_{-m,n} - W_{mn} \hat{Q}_{-m,n} - \hat{V}_{-m,n} Q_{mn} \\ &= V_{mn} \hat{Q}_{-m,n} - \hat{V}_{-m,n} Q_{mn} = O_-(z^{m+n}) \\ &= [\hat{V}_{-m,n} + \hat{W}_{-m,n}] Q_{mn} - W_{mn} \hat{Q}_{-m,n} - \hat{V}_{-m,n} Q_{mn} \\ &= \hat{W}_{-m,n} Q_{mn} - W_{mn} \hat{Q}_{-m,n} = O_+(z^{n-m}). \end{aligned}$$

This completes the proof for the first part of the theorem. The second part has a similar proof. $\square$

We shall call Laurent-Padé table, the table with entries $K_{\mu\nu}, \mu=0, \pm 1, \pm 2, \dots; \nu=0, 1, 2, \dots$. Before we investigate its structure, we consider:

3. The T-table.

We define the T-table as the infinite matrix with (m, n) -th entry T_{mn} , $m = 0, \pm 1, \pm 2, \dots, n = 0, 1, 2, \dots$, where T_{mn} is the Toeplitz determinant defined in section 1. It is known that the corresponding table in the classical Padé case has a characteristic block structure. (see theorem 3.2 of [1]). Let $\text{ord}_+ F(z) = m^+ < \infty$. Then, except for a shift over m^+ rows, the block structure of the T-table is as described and proved in Gragg's theorem 3.2. Similarly with the transformation $z \rightarrow 1/z$, the case $\text{ord}_- F(z) = m^- > -\infty$ is covered by this theorem. In the regions $m > m^+$ and $m < m^-$, the T_{mn} are trivially zero. Therefore we concentrate on $T_{\mu\nu}$ with $m^+ \leq \mu \leq m^-$ and $\nu \geq 0$.

For such (μ, ν) we suppose that $Q_{\mu\nu}, P_{\mu\nu}, V_{\mu\nu}, W_{\mu\nu}$ are solutions of the equations (2.2-5). We shall associate with it a reduced solution as follows. Let $P_{\mu\nu}(z) = z^{k(\mu, \nu)} A_{\mu\nu}$ so that $A_{\mu\nu}$ is a genuine polynomial of degree at most $\ell(\mu, \nu) - k(\mu, \nu)$. Let A/Q be the unique reduced form of $A_{\mu\nu}/Q_{\mu\nu}$ such that $Q(0) = 1$. Set further

$$P(z) = z^{k(\mu, \nu)} A(z), V(z) = \hat{Z}(z)Q(z) + P(z), W(z) = Z(z)Q(z) - P(z). \quad (3.1)$$

We call $S = (P, Q, V, W)$ the reduced solution for equations (2.2-5). Let the greatest common divisor of $A_{\mu\nu}$ and $Q_{\mu\nu}$ producing the reduced form be $z^\lambda D(z)$ with $D(z) = d_0 + d_1 z + \dots + d_k z^k, d_0 d_k \neq 0$ then we clearly have for $S_{\mu\nu} = (P_{\mu\nu}, Q_{\mu\nu}, V_{\mu\nu}, W_{\mu\nu})$ that

$$S_{\mu\nu}(z) = z^\lambda D(z) S(z). \quad (3.2)$$

Now $\text{ord}_- V$ and $\text{ord}_+ W$ will be used to define the top row, respectively the bottom row of the blocks in the T-table. In a classical Padé situation $\text{ord}_- V$ will always be nonnegative, but now $\text{ord}_- V$ can be $-\infty$. In this case, it is impossible to define the bottom row as $\text{ord}_- V + k$ because k must then be ∞ . Finally we note that $\text{ord}_- V = -\infty$ and $\text{ord}_+ W = \infty$ can not occur simultaneously if $F \neq 0$ because $FQ = V+W$.

Theorem 3.1

Let $S = (P, Q, V, W)$ be the reduced solution of equations (2.2-5). Let

$$\text{ord}_- Q = n, \text{ord}_- V = m, \text{ord}_+ W = \hat{m} + n + 1 \quad (3.3)$$

and define $k = \hat{m} - m$.

Then the following statements are true :

(a) $k \geq 0$,

(b) $S = (P, Q, V, W)$ is a reduced solution of equations (2.2-5) if and only if

$$m \leq \mu \leq \hat{m} \quad \text{and} \quad n \leq v \leq n+k. \quad (3.4)$$

For (μ, v) satisfying (3.4) :

(c) $S_{\mu v} = (P_{\mu v}, Q_{\mu v}, V_{\mu v}, W_{\mu v})$ is a solution of (2.2-5) if and only if

$$S_{\mu v} = z^{\lambda_{\mu v}} D(z) S(z)$$

with

$$\begin{aligned} \lambda_{\mu v} &= \max \{0, (\mu - m) + (v - n) - k\} \quad \text{if} \quad m > -\infty \\ &= \max \{0, (\mu - \hat{m}) + (v - n)\} \quad \text{if} \quad \hat{m} < \infty \end{aligned} \quad (3.5)$$

and D a nonzero polynomial of degree at most

$$\kappa_{\mu v} = v - \text{rank} (f_{\mu+i-j})_{i=1, \dots, v; j=0, 1, \dots, v} \quad (3.6)$$

$$\begin{aligned} \text{(d)} \quad T_{\mu n} &\neq 0 & m \leq \mu \leq \hat{m} , \\ T_{mv} &\neq 0 & n \leq v \leq n+k , \quad \text{if} \quad m > -\infty , \\ T_{\hat{m}+1, v} &\neq 0 & n \leq v \leq n+k , \quad \text{if} \quad \hat{m} < \infty , \\ T_{\mu, n+k+1} &\neq 0 & m \leq \mu \leq \hat{m} , \quad \text{if} \quad k < \infty , \\ T_{\mu v} &= 0 & m < \mu \leq \hat{m} \quad \text{and} \quad n < v \leq n+k . \end{aligned}$$

Proof

We give the proof only for $m > -\infty$, in which case $\hat{m} = m+k$. If $m = -\infty$, $\hat{m}$ will be finite and we can rewrite the following proof with m replaced by $\hat{m}-k$.

Like in [1, p14] we easily derive from (3.2) that we must have the relations

$$\begin{aligned} \kappa &\geq 0, \\ \lambda &\geq \max \{0, (\mu - m) + (v - n) - k\}, \\ \kappa + \lambda &\leq \min \{\mu - m, v - n\} . \end{aligned} \quad (3.7)$$

Conversely, let $S = (P, Q, V, W)$ satisfy (2.2-5) with (μ, ν) replaced by (m, n) and suppose $Q(0) = 0$, $\text{ord}_- Q = n$, $\text{ord}_- V = m$ and $\text{ord}_+ W = m+n+k+1$.

Then for λ and κ solutions of (3.7) and for an arbitrary polynomial

$D(z) = d_0 + d_1 z + \dots + d_\kappa z^\kappa$, $d_0 d_\kappa \neq 0$, $S_{\mu\nu}$ given by (3.2) will be a solution of (2.2-5).

Since S is a reduced solution there must exist integers $\kappa, \lambda, \mu, \nu$ satisfying (3.7).

Thus

$$k \geq (\mu - m) + (\nu - n) - \lambda \geq 2\kappa + \lambda \geq 0.$$

This proves (a).

Furthermore (3.7) can only have a solution if

$$\max \{0, (\mu - m) + (\nu - n) - k\} \leq \min \{\mu - m, \nu - n\}, \quad (3.8)$$

which is equivalent with (3.4). Thus also (b) is proved.

To prove (c) we remark that the most general solution of (2.2-5) is obtained by choosing λ as small as possible and κ as large as possible. The minimal λ satisfying (3.7) is given by $\lambda_{\mu\nu}$ in (3.5) and the maximal κ is $\kappa_{\mu\nu}$ as in (3.6) because $\kappa_{\mu\nu} + 1$ is then the number of degrees of freedom we have in the choice of a solution $Q_{\mu\nu}$ for (2.2).

We know that $T_{\mu\nu} = 0$ if and only if (2.2) has a nontrivial solution for $Q_{\mu\nu}$ with $Q_{\mu\nu}(0) = 0$, i.e. with $\lambda > 0$. Because of (3.7) this occurs if and only if $\min \{\mu - m, \nu - n\} > 0$ and (3.8) is satisfied. This is equivalent with statement (d). □

The numbers m, n, k and $\hat{m}$ are related to the unique reduced solution of (2.2-5) and depend for a given F only on μ and ν . Therefore we define with the notation of previous theorem:

$$\begin{aligned} N_{\mu\nu} &= 0 && \text{if } \mu < m^+ \text{ or } \mu > m^- \\ &= n && \text{otherwise,} \\ K_{\mu\nu} &= \infty && \text{if } \mu < m^+ \text{ or } \mu > m^- \\ &= k && \text{otherwise,} \\ M_{\mu\nu} &= -\infty && \text{if } \mu < m^+ \\ &= m^- && \text{if } \mu > m^- \\ &= m && \text{otherwise,} \end{aligned}$$

$$\begin{aligned}\hat{M}_{\mu\nu} &= m^+ - 1 & \text{if } \mu < m^+ \\ &= \infty & \text{if } \mu > m^- \\ &= \hat{m} & \text{otherwise.}\end{aligned}$$

With these we define square blocks of indices:

$B_{\mu\nu} = \{(m,n) \mid M_{\mu\nu} \leq m \leq \hat{M}_{\mu\nu}, N_{\mu\nu} \leq n \leq N_{\mu\nu} + K_{\mu\nu}\}$ and by deleting its first row and first column we obtain

$$\beta_{\mu\nu} = \{(m,n) \mid M_{\mu\nu} < m \leq \hat{M}_{\mu\nu}, N_{\mu\nu} < n \leq N_{\mu\nu} + K_{\mu\nu}\}.$$

Part (d) of previous theorem says that $T_{mn} = 0$ for $(m,n) \in \beta_{\mu\nu}$ and that for all the finite couples (m,n) bordering $\beta_{\mu\nu}$, T_{mn} will be nonzero.

4. The Laurent-Padé table

We consider now the table with entries $K_{\mu\nu}$ as defined in theorem 2.1. It is a simple consequence of theorem 3.1 (c) that $P_{mn}/Q_{mn} = P_{\mu\nu}/Q_{\mu\nu}$ for all $(m,n) \in B_{\mu\nu}$. Also it follows from defining equations (2.2-5) that $\hat{P}_{\mu+1,\nu}/\hat{Q}_{\mu+1,\nu} = P_{\mu\nu}/Q_{\mu\nu}$. Thus we must also have $\hat{P}_{mn}/\hat{Q}_{mn} = \hat{P}_{\mu\nu}/\hat{Q}_{\mu\nu}$ for all (m,n) in the shifted grid $(1,0) + B_{\mu-1,\nu} = \{(p+1,q) \mid (p,q) \in B_{\mu-1,\nu}\}$. The (rectangular) blocks of equal entries in the Laurent-Padé table are found as follows. Produce a plot of the square blocks $B_{\mu\nu}$. Reflect this plot in the line $\mu = -1/2$ and take the intersections of the original and the reflected squares. These are rectangles with equal K_{mn} entries. More precisely we have

Theorem 4.1.

Define $D_{\mu\nu} = B_{\mu\nu} \cap \hat{B}_{\mu\nu}$ with $\hat{B}_{\mu\nu} = \{(m,n) \mid (-m,n) \in (1,0) + B_{\mu-1,\nu}\}$. Then $K_{mn} = K_{\mu\nu}$ for all $(m,n) \in D_{\mu\nu}$.

Proof

This is very simple to prove. By the remark in the beginning of this section we know that $P_{mn}/Q_{mn} = P_{\mu\nu}/Q_{\mu\nu}$ for $(m,n) \in B_{\mu\nu}$ and $\hat{P}_{-m,n}/\hat{Q}_{-m,n} = \hat{P}_{-\mu,\nu}/\hat{Q}_{-\mu,\nu}$ for $(m,n) \in \hat{B}_{\mu\nu}$.

This proves the theorem. □

We remark that not all of these LPF's are LPA's. As in a classical Padé table, only the elements with indices in the left upper triangular part of a singular block, diagonal included, are Padé approximants. Here we have a similar situation: Let $L_{\mu\nu}$ be this left upper part of $B_{\mu\nu}$ and $\hat{L}_{\mu\nu}$ the left upper part of $\hat{B}_{\mu\nu}$. Then we have

Theorem 4.2

For $\mu \geq 0$

$$(\mu, \nu) \in L_{\mu\nu} \text{ iff } \text{ord}_+ (Z - L_+ (P_{\mu\nu}/Q_{\mu\nu})) \geq \mu + \nu + 1 ,$$

$$(\mu, \nu) \in \tilde{L}_{\mu\nu} \text{ iff } \text{ord}_- (\hat{Z} - L_- (\hat{P}_{\mu\nu}/\hat{Q}_{\mu\nu})) \leq -(\mu + \nu + 1)$$

and

$$K_{\mu\nu} \text{ is a LPA iff } (\mu, \nu) \in L_{\mu\nu} \cap \tilde{L}_{\mu\nu}.$$

Proof

Let $S = (P, Q, V, W)$ be the reduced solution for (2.2-5) with $\text{ord}_- V = m$; $\text{ord}_- Q = n$ and $\text{ord}_+ W = \hat{m} + n + 1$.

Then $\text{ord}_+ (Z - L_+ (P_{\mu\nu}/Q_{\mu\nu})) = \text{ord}_+ (Z - L_+ (P/Q)) = \text{ord}_+ (Z Q - P) = \text{ord}_+ W = \hat{m} + n + 1$. Thus $\text{ord}_+ (Z - L_+ (P_{\mu\nu}/Q_{\mu\nu})) \geq \mu + \nu + 1$ iff $\hat{m} + n \geq \mu + \nu$. I.e. $(\mu, \nu) \in L_{\mu\nu}$. The second relation can be proved similarly. Both relations are true iff $K_{\mu\nu}$ is a LPA. $\square$

5. The Chebyshev-Padé table

For the special case that $f_{-k} = f_k$ for all k , we may replace "Laurent-Padé" by "Chebyshev-Padé" in the previous text. It can directly be verified from their definitions that the symmetry of F implies the following properties :

Theorem 5.1

If $F(z) = F(1/z)$, then for all $\mu = 0, \pm 1, \pm 2, \dots, \nu = 0, 1, \dots$:

$$Q_{-\mu, \nu}(z) = z^\nu Q_{\mu\nu}(1/z); V_{-\mu, \nu}(z) = z^\nu \hat{W}_{\mu\nu}(1/z); W_{-\mu, \nu}(z) = z^\nu \hat{V}_{\mu\nu}(1/z) \text{ and}$$

$$P_{-\mu, \nu}(z) = z^\nu \hat{P}_{\mu\nu}(1/z). \text{ Also } T_{-\mu, \nu} = T_{\mu\nu} \text{ and therefore } B_{\mu\nu} = \hat{B}_{\mu\nu}, \text{ so that}$$

$$D_{\mu\nu} = B_{\mu\nu}.$$

This means that the Chebyshev-Padé table is symmetric with respect to $\mu = -1/2$.

The square blocks $D_{\mu\nu}$ define equal entries and in $L_{\mu\nu}$ these entries are Chebyshev-Padé approximants in the strict sense.

6. Conclusion

We derived the block structure for a Laurent-Padé table. It contains rectangular blocks but these become square again in the symmetric case of Chebyshev-Padé. We derived the results in a formal setting. If the f.l.s is a Laurent series of a function convergent in an annular region of the complex plane one should be more careful in the definition of a LPA as was done in [3].

7. References

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