LAURENT SERIES AND THEIR PADE APPROXIMATIONS

By A.Bultheel

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Abstract

Laurent series and Padé approximants related to it are studied. Computations of poles and zeros and convergence of Laurent-Padé approximants are treated for a meromorphic function.

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1. Introduction.

Padé approximants for a formal power series are well known. The extension of this idea to formal Laurent series has led to the notion of Laurent-Padé approximants. In analogy with the paper of W.B.Gragg [GR2], which treats power series, we tried in this paper to give similar results for a Laurent series.

Up to section 11, the approach is a formal algebraic one. Central in the development are the algorithms given in section 3. This is less conventional since the classical approach is by considering recursive relations for a staircase or a diagonal in the Padé table. The algorithms in section 3 give relations between two adjacent rows in the Padé table. The rhombus rules, which are natural extensions of the classical qd relations are derived from the recursions treated in section 3. They can be found in section 7. The different types of Padé approximants, continued fractions and more general Moebius transformations that are related to these recursions are treated in sections 4-6. The denominators of the approximants are orthogonal polynomials with respect to some linear functionals. These are given in section 8. Expressions in terms of Hankel and Toeplitz determinants are given in section 9 together with the interpretation of the algorithms as factorization algorithms. If the formal Laurent series is symmetric then the upper halves of the bi-infinite tables associated with it are essentially replicants of the lower halves as explained in section 10. Section 11 is the last section on formal series. It extends the classical block structure of Padé tables to Laurent-Padé tables.

Sections 12-16 treat convergence theorems if the Laurent series represents a meromorphic function. Poles and zeros of such a function can be recovered from limits of rows and columns in the bi-infinite tables associated with its Laurent series.

In section 17, we give a special case of the previous theory, relating it to the theory of Schur, Levinson, Szegő polynomials, moment problems and digital filtering. These theories are essentially concerned with the Fourier series of a positive function on the unit circle of the complex plane and study the central row in

the Laurent-Padé table for it. It were these ideas that triggered our interest in extending these results to the complete Laurent-Padé table of a formal Laurent series.

Finally section 18 gives some worked examples to show how the algorithms and formulas work in practice.