On the stability of normalized Powell–Sabin B–splines

Jan Maes, Evelyne Vanraes
Paul Dierckx, Adhemar Bultheel

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

On the stability of normalized Powell–Sabin B–splines

Jan Maes, Evelyne Vanraes
Paul Dierckx, Adhemar Bultheel

Department of Computer Science, K.U.Leuven

Abstract

In this paper we show that the normalized Powell–Sabin B–splines form a stable basis for the max norm. The approximation constants depend only on the smallest angle in the underlying triangulation. Because the B–splines refer to the size of the Powell–Sabin triangles, we have that small Powell–Sabin triangles correspond to better approximation constants than big Powell–Sabin triangles. Next, in addition to the max norm, we treat the L_p norm. Here the approximation constants depend also on a fraction proper to the triangulation. Finally, as a special case, we consider the B–spline bases obtained from Powell–Sabin triangles with minimal area and pay extra attention to the approximation constants for the max norm.

On the stability of normalized Powell–Sabin B–splines

Jan Maes, Evelyne Vanraes
Paul Dierckx, Adhemar Bultheel

May 2003

Abstract

In this paper we show that the normalized Powell–Sabin B–splines form a stable basis for the max norm. The approximation constants depend only on the smallest angle in the underlying triangulation. Because the B–splines refer to the size of the Powell–Sabin triangles, we have that small Powell–Sabin triangles correspond to better approximation constants than big Powell–Sabin triangles. Next, in addition to the max norm, we treat the L_p norm. Here the approximation constants depend also on a fraction proper to the triangulation. Finally, as a special case, we consider the B–spline bases obtained from Powell–Sabin triangles with minimal area and pay extra attention to the approximation constants for the max norm.

Keywords: Powell–Sabin splines, stable bases, approximation

AMS(MOS) classification: 41A15, 65D07

1 Introduction

Let Δ be a triangulation of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\delta\Omega$. The polynomial spline space $S_d^r(\Delta)$ is defined as

$$S_d^r(\Delta) := \{s \in C^r(\Omega) : s|_{\mathcal{T}} \in \Pi_d \text{ for all } \mathcal{T} \in \Delta\}, \tag{1}$$

where $d > r \geq 0$ are given integers and Π_d is the linear space of bivariate polynomials of degree $\leq d$. A basis $\{B_i\}_{i=1}^n$ for a spline space S_d^r which satisfies

$$k_1\|c\|_\infty \leq \|\sum_{i=1}^n c_i B^i\|_\infty \leq k_2\|c\|_\infty \tag{2}$$

for all choices of the coefficient vector c, is called a stable basis. Here k_1 and k_2 are constants which depend only on the smallest angle in Δ.

Finding stable bases for spline spaces $S_d^r(\Delta)$ is a non trivial task for $r > 0$, and can only be done for general triangulations Δ when $d \geq 3r + 2$ [4]. In this paper we study C^1 continuous piecewise quadratic splines, with $r = 1$ and $d = 2$. Because there exists no solution for general triangulations, we restrict ourselves to Powell–Sabin (PS) refinements Δ^* of Δ. The corresponding splines are called Powell–Sabin splines. They appear to be very valuable for CAGD applications [10]. Dierckx [1] proposed a stable algorithm to construct a normalized B–spline representation for such a spline space $S_2^1(\Delta^*)$.

In this paper we prove that the normalized B–spline basis for Powell–Sabin splines is a stable basis. We follow a similar approach as in [5], where it is proven that the Bernstein polynomials of degree d on a triangle $\mathcal{T}$ form a stable basis for Π_d. Related work has been done for a Hermite basis for quadratic splines. Upper bounds were derived for the Hermite basis functions and for their first derivatives [8].

The paper is organized as follows. In section 2 we recall some general concepts of polynomials on triangles and we give the definition of the space of Powell–Sabin splines. This section also covers the relevant aspects of the construction of a normalized B–spline basis. Section 3 is devoted to some useful properties of triangulations. The main theorem, which states that the normalized B–spline basis for PS–splines is a stable basis, is established in section 4. Section 5 extends the main theorem to the L_p norm and in section 6 we treat a special case for the max norm, where we assume that all PS–triangles have minimal area.

2   Powell–Sabin splines

2.1   Polynomials on triangles

Consider a non–degenerated triangle $\mathcal{T}(V_1, V_2, V_3)$ in a plane, having vertices V_i with Cartesian coordinates (x_i, y_i), $i = 1, 2, 3$. This triangle will be denoted as the domain triangle. We define the barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ of an arbitrary point $(x, y) \in \mathbb{R}^2$ with respect to $\mathcal{T}$ as the unique solution to the system

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{3}$$

Each polynomial $P_d(x, y) \in \Pi_d$ on $\mathcal{T}$ has a unique representation

$$P_d(x, y) := b_{\mathcal{T}}^d(\tau) = \sum_{|\lambda| = d} b_\lambda B_\lambda^d(\tau), \tag{4}$$

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i \geq 0$ a multi–index of length $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 = d$, and

$$B_\lambda^d(\tau) = \frac{d!}{\lambda_1! \lambda_2! \lambda_3!} \tau_1^{\lambda_1} \tau_2^{\lambda_2} \tau_3^{\lambda_3} \tag{5}$$

the Bernstein–Bézier polynomials on the triangle [2].

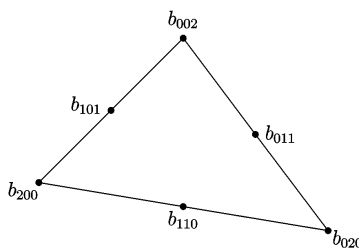

Figure 1: Positions of the Bézier ordinates for $d = 2$.

The coefficients b_λ are called the Bézier ordinates. By associating each ordinate b_λ with the Bézier domain point $(\frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \frac{\lambda_3}{d})$ in the triangle $\mathcal{T}$ we can display this Bernstein–Bézier representation schematically, as in figure 1.

2.2   The linear space $S_2^1(\Delta^*)$

Consider a simply connected subset $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\delta\Omega$. Suppose we have a conforming triangulation Δ of Ω, being constituted of triangles $\mathcal{T}_j$, $j = 1, \ldots, t$, and having

vertices V_i with Cartesian coordinates (x_i, y_i), $i = 1, \ldots, n$. The Powell–Sabin refinement Δ^* of Δ divides each triangle $\mathcal{T}_j$ into six smaller triangles with a common vertex. It can be constructed as follows (see figure 2):
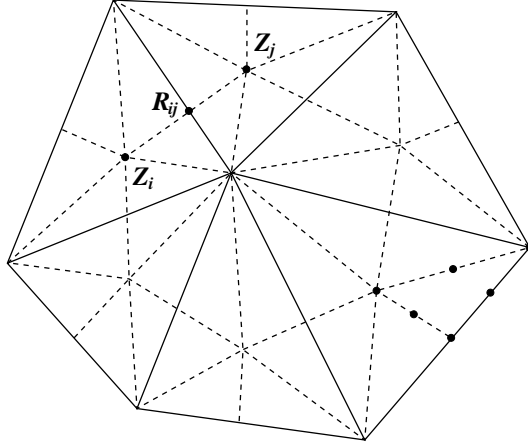

Figure 2: A PS–refinement Δ^*.

1. Choose an interior point Z_j for each triangle $\mathcal{T}_j$, so that if two triangles $\mathcal{T}_i$ and $\mathcal{T}_j$ have a common edge, the line joining Z_i and Z_j intersects this common edge at a point R_{ij} between its vertices. We will choose Z_j as the incenter of triangle $\mathcal{T}_j$.

2. Join the points Z_j to the vertices of $\mathcal{T}_j$.

3. For each edge of $\mathcal{T}_j$

    - which belongs to the boundary $\delta\Omega$, join Z_j to any point on this edge.
    - which is common to a triangle $\mathcal{T}_i$, join Z_j to R_{ij}.

Now we consider the space of piecewise quadratic C^1 continuous polynomials on Ω, the Powell–Sabin splines. This space is denoted by $S_2^1(\Delta^*)$. Each of the $6t$ triangles resulting from the PS–refinement becomes the domain triangle of a quadratic Bernstein–Bézier polynomial, i.e. we choose $d = 2$ in equation (4) and (5), as indicated for one subtriangle in figure 2. Powell and Sabin [7] showed that the following interpolation problem:

$$s(V_k) = f_k, \quad \frac{\partial s}{\partial x}(V_k) = f_{xk}, \quad \frac{\partial s}{\partial y}(V_k) = f_{yk}, \quad k = 1, \ldots, n \tag{6}$$

has a unique solution $s(x, y)$ in $S_2^1(\Delta^*)$. Hence, the dimension of the space $S_2^1(\Delta^*)$ equals $3n$.

2.3 A normalized B–spline representation

Dierckx [1] presented a normalized B–spline representation for Powell–Sabin splines

$$s(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{3} c_{i,j} B_i^j(x, y) \quad , \quad (x, y) \in \Omega \tag{7}$$

where the B–splines form a convex partition of unity on Ω, i.e.

$$B_i^j(x, y) \geq 0 \text{ for all } x, y \in \Omega, \tag{8}$$

3

$$\sum_{i=1}^{n}\sum_{j=1}^{3} B_i^j(x,y) = 1 \text{ for all } x,y \in \Omega. \tag{9}$$

Furthermore these basis functions have local support: $B_i^j(x,y)$ vanishes outside the so–called molecule M_i of vertex V_i, which is the union of all triangles $\mathcal{T}_k$ containing V_i. The molecule number m_i is defined as the number of triangles in the molecule M_i.

The basis functions $B_i^j(x,y)$ can be obtained as follows: find three linearly independent sets $(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})$, $j = 1,2,3$ for each vertex V_i. $B_i^j(x,y)$ is the unique solution of the interpolation problem (6) with $(f_k, f_{xk}, f_{yk}) = (\delta_{ki}\alpha_{i,j}, \delta_{ki}\beta_{i,j}, \delta_{ki}\gamma_{i,j})$, where δ_{ki} is the Kronecker delta.

The sets $(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})$, $j = 1,2,3$ must be determined in such a way that equations (8) and (9) are satisfied. To find appropriate sets $(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})$, $j = 1,2,3$ we use the algorithm from [1].

1. For each vertex $V_i \in \Delta$, find its PS–points. This is a number of particular surrounding Bézier domain points and the vertex V_i itself. Figure 3 shows the PS–points $S, \tilde{S}, S'$ and V_1 for the vertex V_1 in the triangle $\mathcal{T}(V_1, V_2, V_3)$.

2. For each vertex V_i, find a triangle $t_i(Q_{i1}, Q_{i2}, Q_{i3})$ which contains all the PS–points of V_i from all the triangles $\mathcal{T}_k$ in the molecule M_i. These triangles t_i, $i = 1, \ldots, n$ are called PS–triangles and we denote their vertices with $Q_{ij}(X_{ij}, Y_{ij})$. Figure 3 also shows such a PS–triangle t_1.

3. Three linearly independent triplets of real numbers $(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j})$, $j = 1,2,3$ can be derived from the PS–triangle t_i of a vertex V_i as follows:

$$
\begin{aligned}
\alpha_i &= (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}) \text{ are the barycentric coordinates of} \\
&\quad V_i \text{ with respect to } t_i, \\
\beta_i &= (\beta_{i1}, \beta_{i2}, \beta_{i3}) = \left(\frac{Y_{i2} - Y_{i3}}{f}, \frac{Y_{i3} - Y_{i1}}{f}, \frac{Y_{i1} - Y_{i2}}{f} \right), \\
\gamma_i &= (\gamma_{i1}, \gamma_{i2}, \gamma_{i3}) = \left(\frac{X_{i3} - X_{i2}}{f}, \frac{X_{i1} - X_{i3}}{f}, \frac{X_{i2} - X_{i1}}{f} \right),
\end{aligned}
$$

where

$$
f = \begin{vmatrix} X_{i1} & Y_{i1} & 1 \\ X_{i2} & Y_{i2} & 1 \\ X_{i3} & Y_{i3} & 1 \end{vmatrix}.
$$

We have $|\alpha_i| = 1$ and $|\beta_i| = |\gamma_i| = 0$.

A useful consequence is the notion of control triangles. First, we define the PS–control points as

$$C_{i,j}(X_{i,j}, Y_{i,j}, c_{i,j}). \tag{10}$$

For fixed i, they constitute a triangle $T_i(C_{i,1}, C_{i,2}, C_{i,3})$ that is tangent to the surface at $(V_i, s(V_i))$. The projection of the control triangles T_i in the (x,y) plane are the PS–triangles t_i. The area of a PS–triangle t_i equals

$$A_{t_i(Q_{i1}, Q_{i2}, Q_{i3})} = \frac{1}{2|\beta_{i1}\gamma_{i2} - \gamma_{i1}\beta_{i2}|} = \frac{|f|}{2}. \tag{11}$$
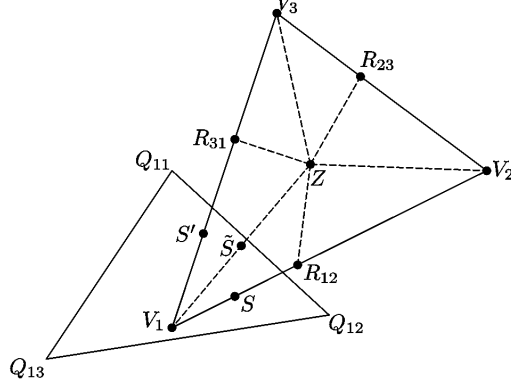
Figure 3: PS–points and PS–triangle.

3 Properties of triangulations and Powell–Sabin refinements

In this section we introduce some useful notation which will be used throughout the remainder of this text, and we collect several properties needed later. Suppose $\mathcal{T}$ is a triangle, then

$$
\begin{aligned}
|\mathcal{T}| &:= & \text{the diameter of the smallest disk containing } \mathcal{T}, \\
\rho_\mathcal{T} &:= & \text{the radius of the largest disk contained in } \mathcal{T}, \\
\theta_\mathcal{T} &:= & \text{the smallest angle in the triangle } \mathcal{T}, \\
A_\mathcal{T} &:= & \text{the area of the triangle } \mathcal{T}.
\end{aligned}
$$

Consider a triangulation Δ of a subset $\Omega \in \mathbb{R}^2$ and its PS–refinement Δ^*. Denote the PS–refinement of triangle $\mathcal{T} \in \Delta$ as $\mathcal{T}^*$. We define

$$
\begin{aligned}
|\mathcal{T}^*| &:= & \min_{\mathcal{T}_{PS} \in \mathcal{T}^*} |\mathcal{T}_{PS}|, \\
\rho_{\mathcal{T}^*} &:= & \min_{\mathcal{T}_{PS} \in \mathcal{T}^*} \rho_{\mathcal{T}_{PS}}, \\
\theta_{\mathcal{T}^*} &:= & \min_{\mathcal{T}_{PS} \in \mathcal{T}^*} \theta_{\mathcal{T}_{PS}}, \\
\theta_\Delta &:= & \text{the smallest angle in the triangulation } \Delta, \\
\theta_{\Delta^*} &:= & \text{the smallest angle in the PS–refinement } \Delta^*, \\
A_\Omega &:= & \text{the area of } \Omega.
\end{aligned}
$$

The following lemmas give estimates of the above quantities.

Lemma 3.1 *Consider a triangle $\mathcal{T}$. Then*

$$
\frac{|\mathcal{T}|}{\rho_\mathcal{T}} \leq \frac{4}{\tan(\theta_\mathcal{T}/2)}.
$$

Proof It is well–known that

$$
\rho_\mathcal{T} = \tan(\theta_\mathcal{T}/2) \cdot \frac{a+b-c}{2},
$$

with a, b and c the side lengths of the triangle. Side length c corresponds to the side opposite to the angle $\theta_\mathcal{T}$, and thus has the smallest value. Denote the longest edge of $\mathcal{T}$ with $e_{\max}$, then the following inequalities hold:

$$
\frac{2}{\tan(\theta_\mathcal{T}/2)} = \frac{a+b-c}{\rho_\mathcal{T}} \geq \frac{|e_{\max}|}{\rho_\mathcal{T}} \geq \frac{|\mathcal{T}|/2}{\rho_\mathcal{T}}.
$$

5

□

The following lemma is due to Lai and Schumaker [6].

Lemma 3.2 *Suppose Δ^* is the Powell–Sabin refinement of a given triangulation Δ. Then $\theta_{\Delta^*} \geq \theta_\Delta \sin(\theta_\Delta)/4$.*

Proof See [6]. □

Lemma 3.3 *Suppose $\mathcal{T}$ is a triangle in Δ with PS–refinement $\mathcal{T}^*$. Denote the longest edge in the PS–refinement $\mathcal{T}^*$ of the triangle $\mathcal{T}$ as $e_{\max}$. Then*

$$\frac{1}{\rho_{\mathcal{T}^*}} \leq \frac{4}{\sin(\theta_{\mathcal{T}^*})^4 \tan(\theta_{\mathcal{T}^*}/2)|e_{\max}|}.$$

Proof Let e and $\tilde{e}$ be two edges of the same triangle $\mathcal{T}_{PS} \in \mathcal{T}^*$. Then

$$\sin(\theta_{\mathcal{T}^*})|e| \leq |\tilde{e}|. \tag{12}$$

Suppose we want to compare two arbitrary edges e_1 and e_2 in $\mathcal{T}^*$. Then there always exists a series of edges in $\mathcal{T}^*$ such that

$$|e_1| \leq \left(\frac{1}{\sin(\theta_{\mathcal{T}^*})}\right)^4 |e_2|. \tag{13}$$

Evidently this equation also holds for the maximum and minimum edge. By Lemma 3.1 the following holds:

$$\frac{|e_{\min}|}{\rho_{\mathcal{T}^*}} \leq \frac{|\mathcal{T}^*|}{\rho_{\mathcal{T}^*}} \leq \frac{4}{\tan(\theta_{\mathcal{T}^*}/2)}. \tag{14}$$

Substitute (13) in (14) to prove the lemma. □

Lemma 3.4 *Consider two triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ in Δ with a common edge. Denote the longest edge in the PS–refinement $\mathcal{T}_1^*$ of the triangle $\mathcal{T}_1$ as $e_{\max}(\mathcal{T}_1^*)$ and the longest edge in the PS–refinement $\mathcal{T}_2^*$ of the triangle $\mathcal{T}_2$ as $e_{\max}(\mathcal{T}_2^*)$. Then*

$$\sin(\theta_{\Delta^*})^4 \leq \frac{|e_{\max}(\mathcal{T}_2^*)|}{|e_{\max}(\mathcal{T}_1^*)|} \leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4.$$

Proof From equation (13) we find that

$$\begin{aligned}
|e_{\max}(\mathcal{T}_1^*)| &\leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4 |\tilde{e}| \\
&\leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4 |e_{\max}(\mathcal{T}_2^*)|
\end{aligned} \tag{15}$$

and likewise

$$|e_{\max}(\mathcal{T}_2^*)| \leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4 |e_{\max}(\mathcal{T}_1^*)| \tag{16}$$

with $\tilde{e}$ a common edge of $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$. Combining equation (15) and (16) yields the result. □

6

4 Stability for the max norm

We will now prove that the basis functions $\{B_i^j(x,y)\}_{i=1\ldots n, j=1,2,3}$, introduced in section 2.3, form a stable basis for $S_2^1(\Delta^*)$, i.e. that there exist constants k_1 and k_2 such that for all choices of the coefficient vector c

$$k_1 \|c\|_\infty \leq \|\sum_{i=1}^{n}\sum_{j=1}^{3} c_{i,j} B_i^j(x,y)\|_\infty \leq k_2 \|c\|_\infty$$

with $\|c\|_\infty := \max_{i,j} |c_{i,j}|$ and $\|s\|_\infty := \max_\Omega |s(x,y)|$.

Before we prove the main theorem, we introduce two lemmas. The first lemma (Lemma 4.1) gives an upper bound for $\|D_x s(x,y)\|_{\infty,\mathcal{T}_{PS}}$ and $\|D_y s(x,y)\|_{\infty,\mathcal{T}_{PS}}$, where $\|D_x s\|_{\infty,\mathcal{T}_{PS}} := \max_{\mathcal{T}_{PS}} |D_x s(x,y)|$ and $\mathcal{T}_{PS}$ is a triangle in the PS–refinement Δ^*. This upper bound will be useful in the proof of Theorem 4.3.

Lemma 4.1 *Suppose $s(x,y) \in S_2^1(\Delta^*)$. Consider a triangle $\mathcal{T}_{PS}$ of the PS-refinement Δ^* of Δ. Then*

$$\|D_x s(x,y)\|_{\infty,\mathcal{T}_{PS}} \leq \frac{12}{\rho_{\mathcal{T}_{PS}}} \|s(x,y)\|_{\infty,\mathcal{T}_{PS}}$$

and

$$\|D_y s(x,y)\|_{\infty,\mathcal{T}_{PS}} \leq \frac{12}{\rho_{\mathcal{T}_{PS}}} \|s(x,y)\|_{\infty,\mathcal{T}_{PS}}.$$

Proof We can write $s(x,y)|_{\mathcal{T}_{PS}}$ in its unique Bézier representation:

$$s(x,y)|_{\mathcal{T}_{PS}} := s(\tau) = \sum_{|\lambda|=2} b_\lambda B_\lambda^2(\tau).$$

Denote the vertices of $\mathcal{T}_{PS}$ as $V_i(x_i, y_i)$, $i = 1,2,3$. Let $u = V_2 - V_1 = (x_2 - x_1, \ y_2 - y_1)$ and $v = V_3 - V_1 = (x_3 - x_1, \ y_3 - y_1)$ define two vectors. Then the derivatives of $s(x,y)|_{\mathcal{T}_{PS}}$ with respect to u respectively v are given by

$$\begin{aligned}
D_u s(\tau) &= (x_2 - x_1)D_x s(\tau) + (y_2 - y_1)D_y s(\tau), \\
D_v s(\tau) &= (x_3 - x_1)D_x s(\tau) + (y_3 - y_1)D_y s(\tau).
\end{aligned}$$

Solving for $D_x s(\tau)$ and $D_y s(\tau)$ gives

$$\begin{aligned}
D_x s(\tau) &= \frac{(y_3 - y_1)D_u s(\tau) - (y_2 - y_1)D_v s(\tau)}{f}, \\
D_y s(\tau) &= \frac{(x_2 - x_1)D_v s(\tau) - (x_3 - x_1)D_u s(\tau)}{f},
\end{aligned}$$

from which we find that

$$\|D_x s(\tau)\|_\infty \leq \frac{|y_3 - y_1|}{2A_{\mathcal{T}_{PS}}} \|D_u s(\tau)\|_\infty + \frac{|y_2 - y_1|}{2A_{\mathcal{T}_{PS}}} \|D_v s(\tau)\|_\infty.$$

The area $A_{\mathcal{T}_{PS}}$ is bounded below by

$$\rho_{\mathcal{T}_{PS}} |y_3 - y_1| \leq A_{\mathcal{T}_{PS}}, \qquad \rho_{\mathcal{T}_{PS}} |y_2 - y_1| \leq A_{\mathcal{T}_{PS}}.$$

Substituting in the previous equation gives

$$\|D_x s(\tau)\|_\infty \leq \frac{1}{2\rho_{\mathcal{T}_{PS}}} (\|D_u s(\tau)\|_\infty + \|D_v s(\tau)\|_\infty).$$

7

The estimate for $\|D_y s(\tau)\|_\infty$ can be established in the same way.

Vector u has barycentric coordinates (-1,1,0). The derivative of $s(\tau)$ with respect to u is given by [3]

$$D_u s(\tau) = 2 \sum_{|\lambda|=1} (-b_{\lambda+\vec{e_1}} + b_{\lambda+\vec{e_2}}) B_\lambda^1(\tau)$$

with $\vec{e_1} = (1,0,0)$ and $\vec{e_2} = (0,1,0)$. We now have

$$\|D_u s(x,y)\|_{\infty,\mathcal{T}_{PS}} \leq 2 \sum_{|\lambda|=1} (2\|b\|_\infty) B_\lambda^1(\tau) = 4\|b\|_\infty.$$

The same reasoning gives an analogous estimate for $\|D_v s(x,y)\|_{\infty,\mathcal{T}_{PS}}$. Combining these two estimates yields

$$\|D_x s(x,y)\|_{\infty,\mathcal{T}_{PS}} \leq \frac{4}{\rho_{\mathcal{T}_{PS}}} \|b\|_\infty$$

and

$$\|D_y s(x,y)\|_{\infty,\mathcal{T}_{PS}} \leq \frac{4}{\rho_{\mathcal{T}_{PS}}} \|b\|_\infty.$$

It suffices to prove that

$$\|b\|_\infty \leq 3\|s(x,y)\|_{\infty,\mathcal{T}_{PS}}.$$

Define

$$\xi := \{\left(\frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2}\right) \mid \lambda_1 + \lambda_2 + \lambda_3 = 2, \ \lambda_i \geq 0\}$$

as the set of Bézier domain points. Then

$$[s(\xi)]_{6\times 1} = [B_\lambda^2(\xi)]_{6\times 6} \cdot [b_\lambda]_{6\times 1}.$$

Since interpolation at the Bézier domain points ξ by polynomials in Π_2 is unique, $[B_\lambda^2(\xi)]_{6\times 6}$ is invertible, and we find

$$
\begin{aligned}
\|b\|_\infty &\leq& \|[B_\lambda^2(\xi)]_{6\times 6}^{-1}\|_\infty \cdot \|[s(\xi)]_{6\times 1}\|_\infty \\
&\leq& \|[B_\lambda^2(\xi)]_{6\times 6}^{-1}\|_\infty \cdot \|s(x,y)\|_{\infty,\mathcal{T}_{PS}}.
\end{aligned}
$$

It can easily be verified that $\|[B_\lambda^2(\xi)]_{6\times 6}^{-1}\|_\infty = 3$. $\square$

The second lemma (Lemma 4.2) deals with the choice of the PS–triangles [9]. Recall figure 3. It is clear that there are infinity many choices for the PS–triangle $t_1(Q_{11}, Q_{12}, Q_{13})$, because the only condition for t_1 to be a valid PS–triangle is that t_1 contains the PS–points V_1, S, $\tilde{S}$ and S'. Also in a general situation there are infinity many triangles that form a valid PS–triangle for a vertex $V_i \in \Delta$. The actual choice of a PS–triangle is important, because, as explained in section 2.3, the B–spline basis functions depend on these PS–triangles. As a logic consequence the approximation constants in (2) will be different for another choice of basis functions or another choice of PS–triangles.

Now, suppose we are given a vertex V_i and its surrounding PS–points. Let C_i be the smallest circle with center V_i that contains all the PS–points and denote its radius as ρ_i. It is clear that an equilateral triangle with barycenter V_i and inradius $k\rho_i$ with $k \geq 1$ is a valid PS–triangle for V_i. It is also clear that for every vertex V_i there exists a constant $K_i \geq 1$ such that the actual PS–triangle t_i is contained in such an equilateral triangle with barycenter V_i and inradius $K_i\rho_i$.

Let $K = \max_i K_i$, then Lemma 4.2 is used in Theorem 4.3 to reduce the dependence of the approximation constants in (2) on the PS–triangles t_i to dependence on the constant K. We mention that a scaling operation on the domain does not change the value of K.

8

Lemma 4.2 *Let an arbitrary vertex $V_i \in \Delta$ be given. Consider the surrounding PS–points of vertex V_i and denote the PS–point with the longest distance to vertex V_i as S. Define C_i as the circle with center V_i and radius $K|SV_i|$ where $K \in \mathbb{R}$ is a constant and $K \geq 1$. It is clear that C_i contains all the PS–points. Define $\mathcal{T}_S \in \Delta^*$ as either one of the two triangles that contains the PS–point S and define $\mathcal{T}_{C_i}$ as the set of equilateral triangles that have C_i as its inscribed circle. Suppose t_i is a valid PS–triangle and suppose t_i is contained in a triangle $\mathcal{T}'_{C_i} \in \mathcal{T}_{C_i}$. Then*

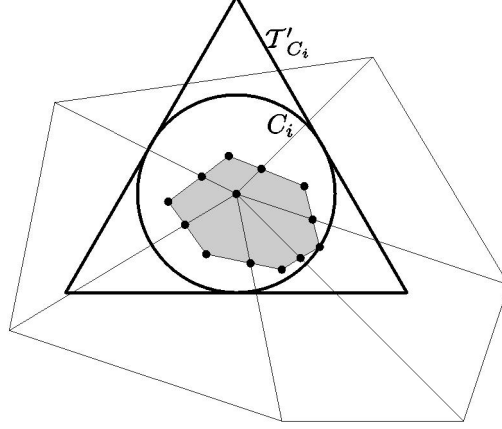$$\frac{|e_{\max}(t_i)|}{|e_{\max}(\mathcal{T}_S)|} \leq \sqrt{3}K.$$

(See figure 4)

Figure 4: Circle C_i and a triangle $\mathcal{T}'_{C_i} \in \mathcal{T}_{C_i}$ for $K = 1$.

Proof Because t_i is contained in $\mathcal{T}'_{C_i}$ it is sufficient to prove that

$$\frac{|e_{\max}(\mathcal{T}'_{C_i})|}{|e_{\max}(\mathcal{T}_S)|} \leq \sqrt{3}K.$$

Clearly,

$$|e_{\max}(\mathcal{T}'_{C_i})| = 2\sqrt{3}K|SV_i|. \tag{17}$$

We also know that

$$|e_{\max}(\mathcal{T}_S)| \geq 2|SV_i|. \tag{18}$$

Combining (17) and (18) proves the lemma. □

Now we come to the main theorem of this paper which states that the normalized B–spline basis functions form a stable basis.

Theorem 4.3 *Consider a triangulation Δ of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\delta\Omega$. Suppose Δ is constituted of triangles $\mathcal{T}_j$, $j = 1, \ldots, t$ which have vertices V_i, $i = 1, \ldots, n$. Define Δ^* as the PS–refinement of Δ. Suppose that there exists a constant K such that every PS–triangle t_i is contained in a triangle $\mathcal{T}'_{C_i} \in \mathcal{T}_{C_i}$ where $\mathcal{T}_{C_i}$ is defined as in Lemma 4.2. Then there exists a constant K_1 depending only on K and on the smallest angle in the underlying triangulation such that for all Powell–Sabin splines $s(x, y) \in S_2^1(\Delta^*)$ in their normalized B–spline representation (7),*

$$\|s(x, y)\|_\infty \leq \|c\|_\infty \leq K_1 \|s(x, y)\|_\infty. \tag{19}$$

9

Proof The left inequality immediately follows from equation (9). We now establish the right inequality. From (7) and the construction of the B–spline basis functions we have

$$
\begin{aligned}
s(V_i) &= \alpha_{i1}c_{i1} + \alpha_{i2}c_{i2} + \alpha_{i3}c_{i3}, \\
D_x s(V_i) &= \beta_{i1}c_{i1} + \beta_{i2}c_{i2} + \beta_{i3}c_{i3}, \\
D_y s(V_i) &= \gamma_{i1}c_{i1} + \gamma_{i2}c_{i2} + \gamma_{i3}c_{i3},
\end{aligned}
$$

or

$$
s = A \cdot c,
$$

$$
\begin{bmatrix} s(V_i) \\ D_x s(V_i) \\ D_y s(V_i) \end{bmatrix} = \begin{bmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \\ \beta_{i1} & \beta_{i2} & \beta_{i3} \\ \gamma_{i1} & \gamma_{i2} & \gamma_{i3} \end{bmatrix} \begin{bmatrix} c_{i1} \\ c_{i2} \\ c_{i3} \end{bmatrix}.
$$

If we take into account that $\alpha_{i3} = 1 - \alpha_{i1} - \alpha_{i2}$, $\beta_{i3} = -\beta_{i1} - \beta_{i2}$ and $\gamma_{i3} = -\gamma_{i1} - \gamma_{i2}$, then we find that the inverse of A is equal to

$$
A^{-1} = \begin{bmatrix} 1 & \eta_{i1} & \tilde{\eta}_{i1} \\ 1 & \eta_{i2} & \tilde{\eta}_{i2} \\ 1 & \eta_{i3} & \tilde{\eta}_{i3} \end{bmatrix},
$$

where

$$
\begin{aligned}
\eta_{ij} &= \frac{\alpha_{i2}\gamma_{i1} - \alpha_{i1}\gamma_{i2} + \delta_{j1}\gamma_{i2} - \delta_{j2}\gamma_{i1}}{\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}}, \\
\tilde{\eta}_{ij} &= \frac{\alpha_{i1}\beta_{i2} - \alpha_{i2}\beta_{i1} - \delta_{j1}\beta_{i2} + \delta_{j2}\beta_{i1}}{\beta_{i1}\gamma_{i2} - \beta_{i2}\gamma_{i1}}.
\end{aligned}
$$

Suppose that $\|c\|_\infty = |c_{ij}|$. Then

$$
\|c\|_\infty = |s(V_i) + \eta_{ij}D_x s(V_i) + \tilde{\eta}_{ij}D_y s(V_i)|.
$$

Define $\mathcal{T}_S$ as in Lemma 4.2. Lemma 4.1 applied to triangle $\mathcal{T}_S$, together with equation (11), yields

$$
\|c\|_\infty \leq \|s(x,y)\|_{\infty,\mathcal{T}_S}\left(1 + |\alpha_{i2}\gamma_{i1} - \alpha_{i1}\gamma_{i2} + \delta_{j1}\gamma_{i2} - \delta_{j2}\gamma_{i1}| \cdot 2A_{t_i} \cdot \frac{12}{\rho_{\mathcal{T}_S}} + \right.
$$

$$
\left. |\alpha_{i1}\beta_{i2} - \alpha_{i2}\beta_{i1} - \delta_{j1}\beta_{i2} + \delta_{j2}\beta_{i1}| \cdot 2A_{t_i} \cdot \frac{12}{\rho_{\mathcal{T}_S}}\right).
$$

If we use the explicit formulas for β_{ij} and γ_{ij}, we get

$$
\|c\|_\infty \leq \|s(x,y)\|_{\infty,\mathcal{T}_S}\left[1 + \frac{24A_{t_i}}{\rho_{\mathcal{T}_S}}\left(\frac{2|X_{i3} - X_{i2}| + 2|X_{i1} - X_{i3}| + 2|Y_{i3} - Y_{i1}| + 2|Y_{i2} - Y_{i3}|}{2A_{t_i}}\right)\right].
$$

Here we have also used the fact that $|\alpha_{ij}| \leq 1$ and $|\delta_{ij}| \leq 1$.
We apply Lemma 3.3 to the equation and find that

$$
\|c\|_\infty \leq \|s(x,y)\|_{\infty,\mathcal{T}_S}\left[1 + \frac{48 \cdot (8|e_{\max}(t_i)|)}{\sin(\theta_{\Delta*})^4 \tan(\theta_{\Delta*}/2)|e_{\max}(\mathcal{T}_S)|}\right].
$$

By Lemma 4.2 and Lemma 3.2 it follows that

$$
\|s(x,y)\|_\infty \leq \|c\|_\infty \leq K_1\|s(x,y)\|_\infty
$$

with

$$
K_1 = \left[1 + \frac{384\sqrt{3}K}{\sin(\theta_\Delta \sin(\theta_\Delta)/4)^4 \tan(\theta_\Delta \sin(\theta_\Delta)/8)}\right].
$$

$\square$

The assumption that there exists a constant K such that every PS–triangle t_i is contained in a triangle $\mathcal{T}'_{C_i} \in \mathcal{T}_{C_i}$ is equivalent with the statement that the area of the PS–triangles has to be bounded. As a consequence we remark that the smaller the PS–triangles the better the approximation constants.

5 Stability for the L_p norm

Theorem 5.1 extends Theorem 4.3 to the L_p norm. As before, Δ is a given triangulation of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\delta\Omega$. A basis $\{B_i\}_{i=1}^n$ for a spline space $S_d^r(\Delta)$ (1) which satisfies

$$k_1 \|c\|_p \leq \frac{\|\sum_{i=1}^n c_i B_i\|_p}{A_\Omega^{1/p}} \leq k_2 \|c\|_p \tag{20}$$

for all choices of the coefficient vector c, is called a stable basis for the L_p norm. Here $\|c\|_p := \left(\sum_{i=1}^n |c_i|^p\right)^{1/p}$ and $\|s\|_p := \left(\int_\Omega |s(x,y)|^p \, dxdy\right)^{1/p}$. We are interested in constants k_1 and k_2 which depend only on the smallest angle in the triangulation Δ.

Theorem 5.1 *Consider a triangulation Δ of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\delta\Omega$. Suppose Δ is constituted of triangles $\mathcal{T}_j$, $j = 1, \ldots, t$ which have vertices V_i, $i = 1, \ldots, n$. Define Δ^* as the PS–refinement of Δ. Suppose that there exists a constant K such that every PS–triangle t_i is contained in a triangle $\mathcal{T}_{C_i}' \in \mathcal{T}_{C_i}$ where $\mathcal{T}_{C_i}$ is defined as in Lemma 4.2. Then there exists a constant K_2 depending only on K and on the smallest angle in the underlying triangulation such that for all Powell–Sabin splines $s(x,y) \in S_2^1(\Delta^*)$ in their normalized B–spline representation (7)*

$$\frac{\min_{\mathcal{T} \in \Delta} A_{\mathcal{T}}^{1/p}}{A_\Omega^{1/p}} \frac{1}{K_2} \|c\|_p \leq \frac{\|s(x,y)\|_p}{A_\Omega^{1/p}} \leq \|c\|_p \tag{21}$$

for $1 \leq p < \infty$.

Proof We have

$$\|s(x,y)\|_p^p = \int_\Omega \left| \sum_{i=1}^n \sum_{j=1}^3 c_{ij} B_i^j(x,y) \right|^p dxdy.$$

Let $1/p + 1/q = 1$, then by Hölder's inequality:

$$
\begin{aligned}
\|s(x,y)\|_p^p &\leq \int_\Omega \|c\|_p^p \left(\sum_{i=1}^n \sum_{j=1}^3 (B_i^j(x,y))^q \right)^{\frac{p}{q}} dxdy \\
&\leq \int_\Omega \|c\|_p^p \left(\sum_{i=1}^n \sum_{j=1}^3 B_i^j(x,y) \right)^{\frac{p}{q}} dxdy \\
&= \|c\|_p^p \cdot A_\Omega.
\end{aligned}
$$

This proves the right inequality.

To prove the left–hand side of (21), we use the fact that all norms on a finite dimensional vector space are equivalent. Consider a triangle $\mathcal{T} \in \Delta$. By mapping $\mathcal{T}$ to the standard simplex $\mathcal{T}_s = \{(x,y)|0 \leq x, y \leq 1, x + y \leq 1\}$, we get that

$$\|s(x,y)\|_{\infty, \mathcal{T}_s} \leq K_3 \|s(x,y)\|_{p, \mathcal{T}_s}.$$

This implies that

$$\|s(x,y)\|_{\infty, \mathcal{T}} \leq \frac{K_3}{A_{\mathcal{T}}^{1/p}} \|s(x,y)\|_{p, \mathcal{T}}.$$

So,

$$\begin{aligned}
\|c\|_p^p &\leq \sum_{\mathcal{T}\in\Delta}\left(\sum_{i|V_i\in\mathcal{T}}\sum_{j=1}^{3}|c_{ij}|^p\right)\\
&\leq \sum_{\mathcal{T}\in\Delta}\left(9\|c\|_{\infty,\mathcal{T}}^p\right)\\
&\leq \sum_{\mathcal{T}\in\Delta}\left(9K_1^p\|s(x,y)\|_{\infty,\mathcal{T}}^p\right)\\
&\leq \sum_{\mathcal{T}\in\Delta}\left(9K_1^p\frac{K_3^p}{A_{\mathcal{T}}}\|s(x,y)\|_{p,\mathcal{T}}^p\right)\\
&\leq 9K_1^p\frac{K_3^p}{\min_{\mathcal{T}\in\Delta}A_{\mathcal{T}}}\sum_{\mathcal{T}\in\Delta}\|s(x,y)\|_{p,\mathcal{T}}^p
\end{aligned}$$

Because $\sum_{\mathcal{T}\in\Delta}\|s(x,y)\|_{p,\mathcal{T}}^p = \|s(x,y)\|_p^p$, we have proven that there exists a constant $K_2 = 9^{1/p}K_1K_3$ such that

$$\|c\|_p \leq \frac{K_2}{\min_{\mathcal{T}\in\Delta}A_{\mathcal{T}}^{1/p}}\|s(x,y)\|_p.$$

$\square$

Theorem 5.1 shows that the basis functions form a stable basis for the L_p norm, but the constant k_1 from (20) contains a factor $\frac{\min_{\mathcal{T}\in\Delta}A_{\mathcal{T}}^{1/p}}{A_{\Omega}^{1/p}}$. Nevertheless our approximation constant is satisfactory because its value does not change by a scaling operation on the domain.

6   Minimal PS–triangles

By using the PS–control points (10) we can interactively change the shape of a PS–spline surface. In order to have a good local control over the spline surface we want the PS–triangles t_i (which are the projection of the control triangles in the (x,y) plane) to be as small as possible. Therefore we are interested in the PS–triangle t_i with the smallest area. To determine these minimal PS–triangles we have to solve a quadratic programming problem as mentioned in [1].

In section 4 we explained that there are infinity many choices for the PS–triangles. In this section we will assume that only the PS–triangles with minimal area are used. For this special case we derive an estimate for the constant K, introduced in section 4, which only depends on the smallest angle in the triangulation Δ.

Theorem 6.1 *Suppose V_i is a vertex in the triangulation Δ with molecule M_i and molecule number m_i and suppose that $V_i \notin \delta\Omega$. Let t_i be the PS–triangle with minimal area that contains all the PS–points of vertex V_i. Denote the PS–point with the largest distance to vertex V_i as S and define C_i as the circle with center V_i and radius $K|SV_i|$. If*

$$K \geq \frac{3}{2\sin(\theta_\Delta\sin(\theta_\Delta)/8)\sin(\theta_\Delta\sin(\theta_\Delta)/4)^{4\pi/\theta_\Delta+4}} \tag{22}$$

then there exists an equilateral triangle $\mathcal{T}_{C_i}'$ with C_i as its inscribed circle that contains t_i (see figure 4).

Proof The PS–points surrounding vertex V_i form a polygon $\mathcal{P}$ with m_i corners. It is clear that the PS–triangle t_i contains this m_i–gon.
Because $V_i \notin \delta\Omega$ we know that V_i lies inside this polygon $\mathcal{P}$. The distance from an edge of the

polygon $\mathcal{P}$ to vertex V_i can be bounded below. If we use the same reasoning as in equation (12), then we get for an arbitrary edge $e_{\mathcal{P}}$ of the polygon $\mathcal{P}$ that

$$|e_{\mathcal{P}} V_i| \geq \frac{1}{2} \min_{e \in M_i^*} |e| \sin(\theta_{\Delta^*}/2).$$

Here, M_i^* is defined as the molecule of vertex V_i in the PS–refinement Δ^* of Δ.
From this inequality we can conclude that the height h_{t_i} of PS–triangle t_i can be bounded below by

$$h_{t_i} \geq \min_{e \in M_i^*} |e| \sin(\theta_{\Delta^*}/2). \tag{23}$$

As in Lemma 4.2, we know that there exists a triangle $\mathcal{T}_S \in M_i^*$ such that (18) is satisfied. Define triangle $\mathcal{T}_i \in \Delta$ as the triangle that contains triangle $\mathcal{T}_S \in \Delta^*$ and define triangle $\mathcal{T}_j \in \Delta$ as the triangle that contains edge $\tilde{e} \in M_i^*$ for which $|\tilde{e}| = \min_{e \in M_i^*} |e|$. Then $\mathcal{T}_i$ and $\mathcal{T}_j$ belong to the same molecule M_i and by Lemma 3.4

$$|e_{\max}(\mathcal{T}_i^*)| \leq \left(\frac{1}{\sin(\theta_{\Delta^*})} \right)^{4 \cdot \frac{m_i}{2}} |e_{\max}(\mathcal{T}_j^*)|,$$

and by (18) and (13)

$$2|SV_i| \leq \left(\frac{1}{\sin(\theta_{\Delta^*})} \right)^{2m_i+4} |e_{\min}(\mathcal{T}_j^*)|,$$

where $\mathcal{T}_i^*$ and $\mathcal{T}_j^*$ are the corresponding PS–refinements. Combining this with (23) gives

$$h_{t_i} \geq 2 \sin(\theta_{\Delta^*}/2) \sin(\theta_{\Delta^*})^{2m_i+4} |SV_i|. \tag{24}$$

Now, assume that for an arbitrary constant $K \geq 1$ we have that

$$|e_{\max}(t_i)| > 2\sqrt{3} K |SV_i|. \tag{25}$$

This means that the PS–triangle t_i is not contained in a triangle $\mathcal{T}_{C_i}'$. The area of PS–triangle t_i can be bounded below by combining equation (24) and (25). We get

$$A_{t_i} > \frac{1}{2} \left(2 \sin(\theta_{\Delta^*}/2) \sin(\theta_{\Delta^*})^{2m_i+4} |SV_i| \right) \left(2\sqrt{3} K |SV_i| \right). \tag{26}$$

The area of a triangle $\mathcal{T}_{C_i}'$ with $K = 1$ is equal to $3\sqrt{3}|SV_i|^2$. If this area is smaller than the right–hand side of (26) we have a contradiction, because t_i is supposed to be the PS–triangle with minimal area. So, we have a contradiction if

$$K \geq \frac{3}{2 \sin(\theta_{\Delta^*}/2) \sin(\theta_{\Delta^*})^{2m_i+4}},$$

or, by Lemma 3.2, if

$$K \geq \frac{3}{2 \sin(\theta_\Delta \sin(\theta_\Delta)/8) \sin(\theta_\Delta \sin(\theta_\Delta)/4)^{2m_i+4}}.$$

It is easy to see that the molecule number m_i can be bounded by

$$m_i \leq \frac{2\pi}{\theta_\Delta}. \tag{27}$$

This proves the theorem. $\square$

Acknowledgements

References

[1] P. Dierckx. On calculating normalized Powell–Sabin B–splines. *CAGD*, 15(3):61–78, 1997.

[2] G. Farin. Triangular Bernstein–Bézier patches. *CAGD*, 3(2):83–128, 1986.

[3] G. Farin. *Curves And Surfaces For Computer Aided Geometric Design: A Practical Guide.* Academic Press, Boston, 4th edition, 1997.

[4] R. Q. Jia. Approximation order from certain spaces of smooth bivariate splines on a three–direction mesh. *Trans. Amer. Math. Soc.*, 295:199–212, 1986.

[5] M. Lai and L. Schumaker. On the approximation power of bivariate splines. *Advances in Comp. Math.*, 9:251–279, 1998.

[6] M. Lai and L. Schumaker. Macro–elements and stable local bases for splines on Powell–Sabin splits. *Math. Comp.*, 72:335–354, 2003.

[7] M. J. D. Powell and M. A. Sabin. Piecewise quadratic approximations on triangles. *ACM Transactions on Mathematical Software*, 3:316–325, 1977.

[8] P. Sablonnière. Error bounds for Hermite interpolation by quadratic splines on an α-triangulation. *IMA Journal of Numerical Analysis*, 7:495–508, 1987.

[9] E. Vanraes, P. Dierckx, and A. Bultheel. On the choice of the PS–triangles. TW Report 353, Department of Computer Science, Katholieke Universiteit Leuven, Belgium, February 2003.

[10] J. Windmolders. *Powell–Sabin splines for Computer Aided Geometric Design.* PhD thesis, Department of Computer Science, Katholieke Universiteit Leuven, Belgium, 2003.