APPLICATIONS OF PADE APPROXIMANTS AND
CONTINUED FRACTIONS IN
SYSTEMS THEORY

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Abstract

A brief introduction to the terminology of Padé approximation is given first. In the next sections we summarize some known applications of Padé approximants and continued fractions in the theory of linear systems, digital filtering and network theory. Relevant references are given.

1. Introduction

In the theory of linear systems, rational approximation plays an important role. Padé approximation is one method to obtain a rational approximation for a (formal) power series. Systems theory is originally developed by engineers while Padé approximation has historically been a research field for mathematician. Some results have therefore been obtained independently by these two groups. It is the intention of this contribution to give some relations between the two fields in the hope that this will raise mutual interest. We shall not give new results but try to give a survey of classical and more recent results and indicate some open problems.

This text is presented to a system engineering audience and we shall therefore start by introducing some notions on Padé approximation and continued fractions. The next sections are then devoted to some system theoretical problems where these approximants and continued fractions can be used.

2. Moebius transforms, continued fractions and Padé approximants

2.1. Moebius transforms

On continued fractions see [1-3].

A linear fractional transform or *Moebius transform* is a map of the form

$$z = t(w) = \frac{a+cw}{b+dw} \quad ad - bc \neq 0$$

If $w = p/q$ and $z = x/y$ then we can write it in the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix}.$$

If $\{t_n\}$ is a sequence of Moebius transforms :

$$t_n(w) = \frac{a_n + c_n w}{b_n + d_n w}, \quad a_n d_n - b_n c_n \neq 0, \quad n = 0, 1, \dots$$

and if $T_0(w) = t_0(w)$, $T_n(w) = T_{n-1}(t_n(w))$, $n = 1, 2, \dots$ we can define A_n, B_n, C_n, D_n by

$$\begin{bmatrix} C_n & A_n \\ D_n & B_n \end{bmatrix} = \begin{bmatrix} c_0 & a_0 \\ d_0 & b_0 \end{bmatrix} \cdot \dots \cdot \begin{bmatrix} c_n & a_n \\ d_n & b_n \end{bmatrix} \quad (2.1)$$

and then

$$T_n(w) = \frac{A_n + C_n w}{B_n + D_n w}.$$

2.2. Continued fractions

A *continued fraction* (CF) can be defined as a special sequence of Moebius transforms. viz.

$$b_0 \neq 0, \quad c_0 = b_0, \quad d_0 = 0 \quad \text{and} \quad a_n \neq 0, \quad c_n = 0, \quad d_n = 1; \quad n = 1, 2, \dots$$

Thus $t_0(w) = \frac{a_0}{b_0} + w$ and $t_n(w) = \frac{a_n}{b_n + w}$, so that (2.1) becomes

$$\begin{bmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{bmatrix} = \begin{bmatrix} b_0 & a_0 \\ 0 & b_0 \end{bmatrix} \begin{bmatrix} 0 & a_1 \\ 1 & b_1 \end{bmatrix} \cdot \dots \cdot \begin{bmatrix} 0 & a_n \\ 1 & b_n \end{bmatrix}. \quad (2.2)$$

$A_n/B_n = T_n(0)$ is called the n -th *convergent* of the CF. Two CFs are called *equivalent* if they have the same convergents. A CF is sometimes denoted as

$$T_n(0) = \frac{a_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n} \quad \text{or} \quad \frac{a_0}{b_0} + \sum_{k=1}^n \frac{a_k}{b_k}.$$

An *even contraction* or the *even part* of this CF is another CF with convergents $T_{2n}(0)$. It is of the form

$$a_0 b_0^{-1} + \frac{a_1 b_2}{a_2 + b_1 b_2} + \sum_{k \geq 2} \frac{-a_{2k-2} b_{2k-2}^{-1} a_{2k-1} b_{2k}}{a_{2k} + (b_{2k-1} + b_{2k-2}^{-1} a_{2k-1}) b_{2k}} \quad (b_{2k} \neq 0).$$

The *odd part* can similarly be defined.

It is not difficult to verify that with the quantities defined in (2.1) we can construct a CF with successive convergents

ly. There exists always a solution for the problem find $A_m(z)$ and $B_n(z)$ such that

$$F(z) B_n(z) - A_m(z) = O_+(m+n+k), \text{ with } k \text{ as large as possible.}$$

The reduced form of $A_m(z)/B_n(z)$ is called the (m/n) Padé approximant (PA) of $F(z)$. It is unique up to a normalizing constant factor.

If $R_{m,n}(z)$ are the (m/n) Padé approximants of $F(z)$ $m, n \geq 0$, we can arrange them in a matrix $[R_{m,n}]_{m,n \geq 0}$ which is called the Padé table of $F(z)$.

An (m/n) PA is called *normal* if the degrees of numerator and denominator are exactly m and n respectively and if also $F(z) B_n(z) - A_m(z) = r_{m+n+1} z^{m+n+1} + O_+(m+n+2)$ with $r_{m+n+1} \neq 0$. The Padé table is called *normal* if all its elements are normal. Several generalizations of the Padé idea exist. We mention some of them. A *formal Newton series* has the form

$$F(z) = c_0 + \sum_{k=1}^{\infty} c_k \prod_{i=1}^k (z - \alpha_i)$$

where α_i is a sequence of (not necessarily distinct) interpolation points. A *Newton-Padé approximant* is then defined as the reduced rational form of type (m/n) such that it "interpolates" at $\alpha_k, k = 1, \dots, m+n+2$ eventually in Hermite sense. If all α_k are distinct this is an ordinary rational interpolation problem, if all $\alpha_k = 0$, this is the ordinary Padé problem. A special case is the *two-point approximant* where the α_k can only be 0 or ∞ . It can be formulated as follows :

Let $F(z) = \sum_{k=0}^{\infty} f_k z^k$ and $\hat{F}(z) = \sum_{k=-\infty}^{-1} \hat{f}_k z^k$ be a pair of fls. The (k,m) two-point PA $A_{m-1}(z)/B_m(z)$ of this pair is as defined in the case of ordinary PA's but now subject to the conditions

$$F(z) A_{m-1}(z) - B_m(z) = O_+(k)$$

and

$$\hat{F}(z) A_{m-1}(z) - B_m(z) = O_-(k-m-1)$$

where k is some integer $0 \leq k \leq 2m$.

This definition can be extended to a pair of the form

$$\left(\sum_{k=-v}^{\infty} f_k z^k ; \sum_{k=-\infty}^{\mu} \hat{f}_k z^k \right), \quad \mu, v \geq 0.$$

The (k,m) approximant is then given by

$$\sum_{k=-v}^{-1} f_k z^k + \sum_{k=0}^{\mu} \hat{f}_k z^k + r_{k,m}(z)$$

where $r_{k,m}(z)$ is the earlier defined 2-point PA of the pair

prefix 2D (two dimensional) for this. Many definitions of 2D PA are possible. Let $F(x,y)$ be a formal power series in x and y and

$$A(x,y) = \sum_{(k,l) \in A} a_{kl} x^k y^l, \quad B(x,y) = \sum_{(k,l) \in B} b_{kl} x^k y^l$$

polynomials such that

$$\Pi_C [F(x,y) B(x,y) - A(x,y)] = 0 \quad (C \subset \mathbb{Z}_+ \times \mathbb{Z}_+)$$

$$\text{with } \Pi_C \left[\sum_{kl} g_{kl} x^k y^l \right] = \sum_{(k,l) \in C} g_{kl} x^k y^l$$

then we can call $A(x,y)/B(x,y)$ up to reduction an (A,B,C) 2D-PA of $F(x,y)$ if C contains as many pairs as possible. With different choices of (A,B,C) correspond different definitions. If certain properties of a 1D PA have to be maintained, conditions are to be imposed on the A,B,C grids allowed. E.g. it is natural to have $A \subset C$, $\# C \geq \# A + \# B$ while A,B and C contain elements "not far from $(0,0)$ ". Typical choices are rectangular or triangular grids in the first quadrant of the (k,l) plane.

See e.g. [28-31] and the references therein. [31] gives also a comparison for different definitions.

Another generalization of all the previous theory is possible if the coefficients in the formal series are matrices, square or rectangular. The square case has been studied to some extent but rectangular series and approximants are still poorly documented in the literature. We shall come back to this at the appropriate moments.

2.4. Padé approximants and continued fractions

Padé approximants and continued fractions are closely related.

Let $P(z)$ and $Q(z)$ be numerator and denominator of the (m/n) PA of $F(z)$, and $R(z) = F(z) Q(z) - P(z)$ the corresponding *residual*. We call the (m/n) and (k/l) PA's *adjacent* if $\max(|m-k|, |n-l|) = 1$. In a normal Padé table we can give recursive relations between numerator, denominator and residual of three adjacent approximants. This means that if r_1 and r_2 are adjacent PA's and r_2 and r_3 are adjacent then we can give a recursive relation between numerators, denominators and residuals of the form

$$S_3(z) = b(z) S_2(z) + a(z) S_1(z)$$

where S stands for P, Q or R and $a(z)$ and $b(z)$ are simple polynomials of at most degree 2. This means that if we walk through the Padé table and compute the approximants by the above recursion, we can construct a CF with successive convergents

quotient-difference rhombus relations [10,15,24]. The *qd-table* constructed by the Rutishauser *qd* algorithm [10,32] is a table containing numbers related to the a_k , b_k numbers in the CF of the staircase and sawteeth paths for all the elements in the Padé table. Type III algorithms are interesting because the limits of rows and columns in the *qd* table give information about the zeros and poles of the functions approximated [10].

For Laurent-Padé approximation, the algorithms are very similar, at least in the normal case. The best known type I algorithm in this context is due to Levinson [25] and the dual type II algorithm is named after Schur [26]. See also [27]. Type III algorithms for LPA are considered e.g. in [1]. For the 2D problem, similar algorithms exist. They usually reduce to a block Hankel or Toeplitz system that can be treated like in the 1D-matrix case [28-30].

3. Linear systems and Padé approximation

We shall consider in what follows, linear, lumped time invariant causal systems. Most of the time they will be discrete. They can be single-input-single-output (SISO) or multi-input-multi-output (MIMO).

A possible difficulty in relating Padé results and system theoretic results is the fact that mathematicians usually work with formal power series in z , while engineers use power series in z^{-1} . A rational transfer function is however written as a ratio of two polynomials in z . If it is constructed as a PA of a power series in z^{-1} , it is a rational expression in z^{-1} , but by multiplication with a suitable power of z , in numerator and denominator, it can be transformed into a ratio of two polynomials in z . By this transformation we obtain a rational expression of type (n/n) , or if the constant term in the power series is zero, of type $(n-1/n)$. For this reason the main diagonal and the adjacent superdiagonal elements of the Padé table are of special interest in systems theory.

4. Stability analysis

Stability of a system is connected with the positions of the poles of a rational function. They should be in the left half plane (continuous case) or inside the unit disc (discrete case). Algorithms of Routh-Hurwitz and Schur-Cohn give possibilities to compute numbers from which we can decide on the stability properties by inspection.

Let $p(z)$ be a polynomial with zeros z_i . Separate even and odd powers by $p(z) = t(z^2) + z s(z^2)$, then $p(z)$ is stable ($\operatorname{Re}(z_i) < 0$) iff $r(z) = s(z)/t(z)$ has a finite Stieltjes fraction expansion. I.e.

$$r(z) = \frac{\rho_1}{1} + \sum_{k=2}^n \frac{\rho_k z}{1} \quad \text{with} \quad \rho_k > 0$$

and because of the transformation from z^{-1} to z , we could walk along the first superdiagonal as well or a staircase alternating between those two. Algorithms in the style of Euclid's or Berlekamp-Massey's algorithm give here the proper answer. The solution of the partial identification or realization problem may not be unique even if it is minimal. This can easily be derived from the block structure of the Padé table.

Fig. 1

Fig. 1 represents a singular block of equal PA's in the Padé table of $H(z)$ with left top element the (k/n) PA. The point indexed as (1) is the (n/n) PA, (2) is $(n/n+1)$ PA etc... (1) is completely defined by the sequence $\{h_k\}_0^{2n}$. By inspection of the data $h_{2n}, \dots, h_{2n+d}$ we can verify that the corresponding PA's (1) ... (d) are all in the same singular block. If h_{2n+d} is the last of the given Markov parameters we can't go any further. However it is clear from fig. 1 that, whatever the next elements h_k will be, we shall need at least d more elements to leave the current singular block (remember that it is always square). These can be seen as parameters so that we have a manifold of partial approximants all of the same minimal degree for the sequence $\{h_k\}_0^{2n+d}$ [33]. In [33] the connection between PA's, C F's and the P-fraction algorithm of Magnus and the Berlekamp-Massey algorithm in the context of partial identification and partial realization has been worked out for the scalar case. It is also shown how the parameters of the C F expansion can be used to write down a partial minimal realization triple $\{A, B, C\}$ without further calculations. See also [34]. Such algorithms may be extended to the square matrix case: see [35] under normality conditions and [16] for the general case. The non-square matrix case seems not be completely understood in this context. See [33-34, 45-46]. The available theory for the 2D case is even more confusing since a multitude of possible definitions exist. Problems like non-uniqueness, unknown behavior in a non-normal situation etc. leave many questions unanswered for the moment.

5.2. Stability

Let's come back to the problem of stability in the simplest (1D-scalar) situation. Since stability is an important property for an approximant to be of any value, it has received considerable attention in the literature. A PA of a stable transfer function need not be stable in general. Several remedies have been proposed to cure this situation. The possible degrees of freedom of the minimal partial identification problem can be used in some cases to make the approximant stable.

$$\mu_k = \int t^k h(t) dt \text{ so that } H(z) = \int e^{-zt} h(t) dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k z^k$$

with $h(t)$ the impulse response of the system.

Instead of a PA for $H(z)$ from its expansion around $z = \infty$ we could take a PA for $H(z)$ from its expansion around $z = 0$. This is called *moment matching*. A two point PA matching some Markov parameters and some modified moments is called a *mixed approximant*. A moment matching approximant approximates better the steady state response while a PA at $z = \infty$ is better to approximate the transient behavior of the system. A mixed approximant tries to balance between both. For a discrete time system moment matching corresponds to PA for an expansion around $z = 1$ [42]. Of course problems of the same type as for PA at $z = \infty$ occur. They can be treated as before. Associated with these three types of approximants are some matrix C F's (square matrix case) known in engineering literature as the *matrix Cauer forms*.

$$\text{1st kind : } H(z) = (H'_1 z + (H'_2 + (H'_3 z + (H'_4 + (\dots)^{-1})^{-1})^{-1})^{-1})^{-1}$$

$$\text{2nd kind : } H(z) = (H_1 + (H_2 z^{-1} + (H_3 + (H_4 z^{-1} + (\dots)^{-1})^{-1})^{-1})^{-1})^{-1}$$

$$\text{3rd kind : } H(z) = (K_1 + K'_1 z + (K_2 z^{-1} + K'_2 + (K_3 + K'_3 z + (K_4 z^{-1} + K'_4 + (\dots)^{-1})^{-1})^{-1})^{-1})^{-1}$$

The first is associated with a PA at $z = \infty$ (equivalent form of staircase C F of table 1 with $z \rightarrow z^{-1}$), the second with a PA at $z = 0$ i.e. moment matching (equivalent form of staircase C F), the third gives a mixed approximant (= equivalent form of row C F). Remark that these C F's exist in this form for a normal Padé table and the nonnormal case seems to be overlooked in systems theory with the exception of the Berlekamp-Massey algorithm. However this case has been treated by Padéists.

Some references [43] (moment matching multivariable), [44] (mixed approximant, single variable), [45-46] (mixed approximant, multivariable, nonsquare). For 2D systems, C F's of the above form were obtained in [47-48].

7. Identification from the squared magnitude spectrum

Instead of the impulse response or transfer function or moments, one could use second order information like the autocorrelation function r_k or power spectrum $R(z)$ of the system. Under suitable conditions the relation for a discrete time system is :

$$R(z) = H(z) H_{\star}(z) = \sum_{k=-\infty}^{\infty} r_k z^{-k}.$$

$H_{\star}(z) = H^H(1/z)$ where the superscript H means conjugate transpose. Clearly

The type II dual of the Levinson algorithm is the Schur algorithm. It does not evaluate the denominators $Q_n(z)$ explicitly but performs bilinear transforms of the same type on the residuals to obtain the reflection coefficients directly. In a way it is the inverse of the Schur-Cohn stability test algorithm. In computing row 0 in the Laurent-Padé table we are very lucky because theoretically we shall have no problems of traversing a singular block. This is a consequence of the positivity of the r_k sequence. Type III algorithms compute more of the table than this row 0, and thus they may have to cope with the block structure. This and their known numerical instability make them unsuited for practical use. It is however useful to study them theoretically in this context because they give information about the transmission zeros α_k . [50,51].

Methods of Chebyshev-Padé approximation can also be used for continuous time systems [52,53].

Up to now the approximant had a trivial numerator. In filtering terms an *Autoregressive* (AR) filter is constructed. If the transmission zeros α_k obtained by a type III algorithm or in any other way are not all chosen at the origin we obtain a general *Autoregressive-Movingaverage* (ARMA) filter. The Nevanlinna-Pick algorithm is an interpolating version of the Schur algorithm. It constructs like the Schur algorithm a CF for the S function $\Gamma(z)$, now interpolating in the points α_k , $k = 1, 2, \dots$. Again, the denominator will automatically be stable. The generalization of the recursion (7.1) is characterized by the recursion matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \zeta_k(z) \end{bmatrix} \begin{bmatrix} 1 & -\bar{\rho}_k \\ -\bar{\rho}_k & 1 \end{bmatrix}$$

with $\zeta_k = (z - \alpha_k)(1 - \bar{\alpha}_k z)^{-1}$. These recursions are related to those appearing in the Darlington synthesis procedure of network theory [54] (scalar case), [55] (matrix case). Instead of choosing the numerator a priori, we could first construct an AR approximant with the above methods and afterwards find a numerator with another approximation method [56].

Some final remarks : the predictor polynomial was introduced here as the denominator of a (0/n) LPA. It can also be introduced as the denominator of an ordinary PA for $R(z)$ after shifting an truncation [50]. This idea is also used in an operator theoretic approach of [57]. Some ideas of minimal partial realization under positivity conditions are given in [58]. The approximants of this section are related to Fourier approximation and orthogonal polynomials and are therefore not only approximants in a Padé sense but are also least squares approximants in a certain way.

8. The realization of transfer functions, digital filters and m-port networks

The construction of a rational approximant via type II algorithms may seem an indi-

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Appendix A

For the convenience of mathematicians who are not familiar with a systems theoretic vocabulary, we give in this appendix a collection of definitions. Of course we are far from complete. E.g. we shall not include the important stochastic description of systems and also the multidimensional case is only briefly mentioned at the end.

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$$y(k) = \sum_i h_{k-i} u_i = \sum_i h_i u_{k-i}.$$

Thus for an arbitrary input $u(k)$, the output is given by a convolution

$$\{y(k)\} = \{h(k)\} \star \{u(k)\}. \quad (2)$$

Instead of working with sequences, it is often much more convenient to work with functions. Therefore associate with a sequence $\{f(k)\}$ the formal power series

$$F(z) = \sum_k f(k) z^{-k}. \quad (3)$$

$F(z)$ is called the z -transform of $\{f(k)\}$: $F(z) = Z[f(k)]$. Under certain conditions, this $F(z)$ can represent a function. $H(z) = Z[h(k)]$ is called the *transfer function* of the system. Remark that in this transform domain relation (2) goes over into

$$Y(z) = H(z) U(z). \quad (4)$$

Remark that a multiplication with z^{-1} causes a shift in the time domain: $z^{-1} F(z) = Z[f(k+1)] \triangleq Z[S f(k)]$. Therefore z^{-1} or S is called a *delay*. If $|z| = 1$ is in the convergence region of series (3), then we may set $z = e^{j\omega}$ so that

$$F(e^{j\omega}) = \sum_k f(k) e^{-jk\omega}$$

becomes a Fourier series.

The continuous analog of the impulse response and transfer function are introduced as follows. Give as an input to a continuous time system the generalized function $\delta(t)$.

$$\int f(\tau) \delta(t-\tau) d\tau = f(t).$$

Then the corresponding output $h(t)$ is called the *impulse response*.

Because of causality again $h(t) = 0$ for $t < 0$. A general output is a convolution: $y = h \star u$ viz:

$$y(t) = \int_{-\infty}^t h(t-\tau) u(\tau) d\tau = \int_0^{\infty} h(\tau) u(t-\tau) d\tau. \quad (5)$$

$$\begin{aligned} x(k+1) &= A x(k) + B u(k) & x(0) &= x_0 \\ y(k) &= C x(k) + D u(k) & k &> 0 \end{aligned} \quad (8a)$$

or

$$\begin{aligned} x(t) &= A x(t) + B u(t) & x(0) &= x_0 \\ y(t) &= C x(t) + D u(t) & t &\geq 0 \end{aligned} \quad (8b)$$

$$A \in \mathbb{C}^{n \times n}, \quad B \in \mathbb{C}^{n \times p}, \quad C \in \mathbb{C}^{p \times n}, \quad D \in \mathbb{C}^{p \times m}.$$

A is called the *state-transition matrix* and the equations are the *state-space equations*.

A solution for these equations is given by

$$y(k) = C A^k x_0 + \sum_{j=0}^{k-1} C A^{k-j-1} B u(j) + D u(k) \quad k > 0$$

or

$$y(t) = C e^{At} x_0 + \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau + D u(t) \quad t \geq 0.$$

As a special case, the impulse response with initial state $x_0 = 0$ and $D = 0$ is given by

$$h(k) = C A^{k-1} B \quad k > 0 \quad (9a)$$

or

$$h(t) = C e^{At} B = \sum_{i=0}^{\infty} C A^i B \frac{t^i}{i!} \quad (9b)$$

The corresponding transfer function is for both cases

$$H(z) = C(zI - A)^{-1} B. \quad (10)$$

The state vector should completely characterise the system without redundancies. In other words the input should influence each state independently (this is called *complete reachability* : CR) and the output in turn should be influenced by each state independently (this is called *complete observability* : CO). We shall not give the precise definitions

and set $z = e^s$, then like in the above calculations you obtain

$$H(z) = \sum_k \frac{(-1)^k s^k}{k!} \mu_k.$$

This is an expansion of $H(z)$ around $s = 0$ thus we have

$$\mu_k = (-1)^k \left(z \frac{d}{dz} \right)^k H(z) \Big|_{z=1}.$$

Thus μ_k is expressible as a linear combination of $\left(\frac{d}{dz} \right)^i H(z) \Big|_{z=1}$, $i = 1, 2, \dots, k$.

Now let $A' = A - \alpha I$ (and suppose $A - \alpha I$ is nonsingular), then the substitution $\zeta = z-1$ in (10) gives

$$H(\zeta) = C(\zeta I - A')^{-1} B = \sum_{k=1}^{\infty} h'(k) \zeta^{-k} = - \sum_{k=1}^{\infty} m'(k) \zeta^{k-1}$$

$$\text{where } h'(k) = C(A')^{k-1} B \text{ and } m'(k) = C(A')^{-k} B = - \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} H(z) \Big|_{z=\alpha}$$

From this we may conclude that if the system $\{F, B, C\}$ satisfies

$$C(F')^k B = C(A')^k B \quad k = 0, 1, \dots, p \quad \text{where } F' = F - \alpha I$$

then also $C F^k B = C A^k B \quad k = 0, 1, \dots, p$, thus it has the same Markov parameters as $\{A, B, C\}$.

But also if $C(F')^{-k} B = C(A')^{-k} B \quad k = 1, 2, \dots, q$, then the i -th derivative of the corresponding transfer functions at $z = \alpha$ are the same for $i = 1, 2, \dots, q$, and thus the corresponding time moments match if $\alpha = 1$.

We have hereby the following

Theorem

If $H_1(z)$ and $H_2(z)$ are the transfer functions of two systems and

$$H_1^{(i)}(z) \Big|_{z=\infty} = H_2^{(i)}(z) \Big|_{z=\infty}, \quad i = 1, \dots, p,$$

then they have the same Markov parameters $h(k)$, $k = 1, \dots, p$.

If

$$H_1^{(i)}(z) \Big|_{z=\alpha} = H_2^{(i)}(z) \Big|_{z=\alpha}, \quad i = 0, 1, \dots, q,$$

The problem of *model reduction* can be solved via partial realization. The problem is given a system with a rational transfer function of degree n , to find a simpler system, i.e. one with a transfer function of lower degree which approximates the given system as good as possible with respect to some approximation criterion. Clearly one way to obtain a reduced model is solving a partial realization problem. Instead of prescribing only Markov parameters for the system so that the approximating system will model the transient behavior of the system, we could instead prescribe values for

$$C A^k B \quad \text{for} \quad k = -p, -p+1, \dots, q$$

so that the first $q-1$ Markov parameters and the first p time moments are matched. Thus the transient as well as the steady state behavior of the system are modelled.

An important property in the analysis of systems is their *stability*. It is the property that only a bounded output is expected when a bounded input is applied. Many notions of stability exist. We shall introduce some of them.

A system $\{A, B, C\}$ is *asymptotically stable* (AS) if for $u(t) = 0 \quad \forall t \geq 0$ for a continuous system or $\forall t > 0$ for a discrete system we have for every $x(0)$ that $\lim_{k \rightarrow \infty} x(t) = 0$.

It can be shown that a necessary and sufficient condition for AS is :
All the eigenvalues of A

have a negative real part (continuous case)
are bounded by 1 in modulus (discrete case).

Since these eigenvalues are exactly the poles of the transfer function, we can give these conditions in terms an external description if we replace the eigenvalues of A by the poles of the transfer function.

A weaker notion of stability is *Bounded-Input-Bounded-Output Stability* (BIBOS). The definition of BIBOS is the same for continuous and discrete systems. We call the system BIBOS if for $x(0) = 0$ and for given $v > 0$ there exists $q > 0$, such that $|u(t)| \leq v$ for $t \geq 0$ implies $|y(t)| \leq q$ for $t \geq 0$. This is equivalent with

$$\int_0^{\infty} |h(t)| dt < \infty \quad \text{or} \quad \sum_{k=1}^{\infty} |h(k)| < \infty$$

where $h(t)$ or $h(k)$ are the impulse responses of the continuous or discrete system.

Clearly at this point Fourier transforms come into the picture. The Fourier transform is heavily used in *signal processing*. A signal is nothing but a function of time discrete or continuous and thus a system is a transformation of a signal into another signal. The system is then called a *filter*. A discrete signal is often a sampled version of a continuous (analog) signal. For a discrete input-output relation the filter is called a *digital filter*.

We use a superscript H to mean a complex conjugate transpose.

If $f(t)$ is a signal then its *energy* is

$$\int_{-\infty}^{\infty} f^H(t) f(t) dt \quad \text{or} \quad \sum_{-\infty}^{\infty} f^H(k) f(k).$$

Suppose this energy is finite.

The *autocorrelation function* is then

$$r(\tau) = \int f(t+\tau) f^H(t) dt \quad \text{or} \quad r(i) = \sum_k f(k+i) f^H(k).$$

The *energy spectrum* is

$$\begin{aligned} R(z) &= \mathcal{L}[r(\tau)] & \text{or} & & R(z) &= \mathcal{Z}[r(i)] \\ &= F(z) F_{\star}(z) & & & &= F(z) F_{\star}(z) \end{aligned}$$

$$\text{where } F(z) = \mathcal{L}[f(t)] \quad \text{or} \quad F(z) = \mathcal{Z}[f(k)]$$

$$\text{and } F_{\star}(z) = F^H(-z) \quad F_{\star}(z) = F^H(1/z)$$

Remark that $R(z) \geq 0$ or $R(z) \geq 0$ for
for z on the imaginary axis z on the unit circle.

For signals with infinite energy we can introduce the notion of power. The *power* for a continuous signal is

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f^H(t) f(t) dt$$

the autocorrelation function is then

$$r(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t+\tau) f^H(t) dt$$

$$y(n) = \sum_{k=1}^N a_k y(n-k) + \sum_{k=0}^M b_k u(n-k).$$

This can be represented by

z^{-1} represents a shift or delay and if no transmittance is indicated, then it is supposed to be 1.

If the number of source and sink nodes is the same we can couple them in pairs. Such a couple is called a *port*. If there are m such ports, the network is called an m -port. The above example is a 1-port. A system with as many inputs as outputs can be seen as an m -port.

An m -port N is depicted as :

Originally, an m -port is a physical device consisting of a collection of circuit elements such as resistors, capacitors, inductors, gyrators To each port is associated a voltage and a current, (continuous in time). Thus to an m -port is associated an m -vector of voltages and an m -vector of currents. Denote the Laplace transforms of these vectors by $V(z)$ and $I(z)$ respectively, then the network imposes a relation of the form

$$V(z) = Z(z)I(z).$$

$Z(z)$ is called the *impedance matrix*. Conversely if we take $V(z)$ as an independent variable; then an *admittance matrix* $Y(z)$ may exist such that

$$I(z) = Y(z)V(z).$$

The term *immittance* is used to denote either an impedance or an admittance. To an m -port we can connect a *voltage (current) source*. It delivers a

An immittance matrix $X(z)$ of a passive network is *positive real* i.e.

- (i) All elements of $X(z)$ are analytic in $\text{Re}(z) > 0$
- (ii) $X(z)$ is real for $z > 0$
- (iii) $X^H(z) + X(z) \geq 0$ for $\text{Re}(z) > 0$.

If the network is also lossless then

- (iv) $X^H(z) + X(z) = 0$ for $\text{Re}(z) = 0$.

When 2m-ports are used in *cascade connection* like

It is often convenient not to use the scattering matrix description but use a relation of the form

$$\begin{bmatrix} A_2(z) \\ B_2(z) \end{bmatrix} = \Theta_1(z) \begin{bmatrix} A_1(z) \\ B_1(z) \end{bmatrix}$$

$\Theta_1(z)$ is called the *chain scattering matrix*. These have the pleasant property that

$$\begin{bmatrix} A_2(z) \\ B_2(z) \end{bmatrix} = \Theta_1(z) \begin{bmatrix} A_1(z) \\ B_1(z) \end{bmatrix}, \quad \begin{bmatrix} A_3(z) \\ B_3(z) \end{bmatrix} = \Theta_2(z) \begin{bmatrix} A_2(z) \\ B_2(z) \end{bmatrix} \Rightarrow \begin{bmatrix} A_3(z) \\ B_3(z) \end{bmatrix} = \Theta_2(z) \Theta_1(z) \begin{bmatrix} A_1(z) \\ B_1(z) \end{bmatrix}$$

Passivity on losslessness implies a skew bounded real property for Θ , i.e.

$$\Theta(z)J\Theta^H(z) \leq J \quad \text{for } \text{Re}(z) > 0$$

and

$$\Theta(z)J\Theta^H(z) = J \quad \text{for } \text{Re}(z) = 0.$$

where $J = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}$.

Of course we could also consider voltage, current ... in discrete time. In that case replace the integral by a summation, the Laplace transform by z-transform in the bounded real and positive real conditions. The right half plane is replaced by the exterior of the unit disc and the imaginary axis by the unit circle.

APPENDIX B

In this appendix we give a list of references to papers and books from the literature of systems theory, digital filtering, information theory and network theory. In these references continued fractions and Padé approximants are more or less explicitly used. If anybody has corrections or additional references we would appreciate very much if he would send us his informations.

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