Compactly Supported Powell–Sabin Spline Multiwavelets in Sobolev Spaces

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Report TW 428, May 2005

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Abstract

In this paper we construct Powell–Sabin spline multiwavelets on the hexagonal lattice in a shift-invariant setting. This allows us to use Fourier techniques to study the range of the smoothness parameter s for which the wavelet basis is a Riesz basis in the Sobolev space $H^s(\mathbb{R}^2)$, and we find that $0.431898 < s < 5/2$. For those s, discretizations of H^s-elliptic problems with respect to the wavelet basis lead to uniformly well-conditioned stiffness matrices, resulting in an asymptotically optimal preconditioning method.

Keywords : Powell–Sabin splines, biorthogonal wavelets, lifting
AMS(MOS) Classification : 65T60, 65D07, 41A15

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Abstract

In this paper we construct Powell–Sabin spline multiwavelets on the hexagonal lattice in a shift-invariant setting. This allows us to use Fourier techniques to study the range of the smoothness parameter s for which the wavelet basis is a Riesz basis in the Sobolev space $H^s(\mathbb{R}^2)$, and we find that $0.431898 < s < 5/2$. For those s, discretizations of H^s-elliptic problems with respect to the wavelet basis lead to uniformly well-conditioned stiffness matrices, resulting in an asymptotically optimal preconditioning method.

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1 The primal scaling vector on $\mathbb{R}^2$

Consider the hexagonal lattice Δ in $\mathbb{R}^2$ which is defined as the image of $\mathbb{Z}^2$ by a linear transformation corresponding to the matrix

$$\Gamma = \begin{bmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix},$$

and let Δ^* be its refinement by drawing in the additional grid lines $y = l$, $y = \frac{\sqrt{3}}{3}(x+m)$, and $y = -\frac{\sqrt{3}}{3}(x+n)$, $l, m, n \in \mathbb{Z}$. In fact, Δ^* is the Powell–Sabin 6-split of Δ, see Figure 1. Define $S_2^1(\Delta^*)$ as the space of real-valued functions in $C^1(\mathbb{R}^2)$ whose restrictions on each triangle of the triangulation Δ^* are bivariate quadratic polynomials. Then each function $\phi \in S_2^1(\Delta^*)$ is called a uniform Powell–Sabin (PS) spline.

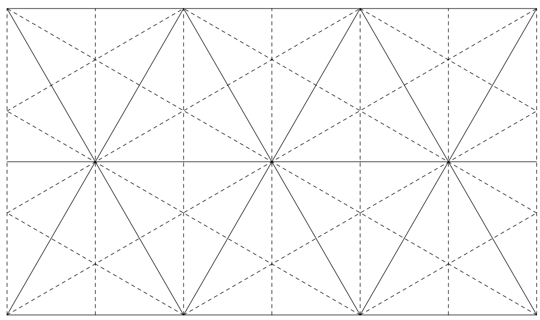

Figure 1: Hexagonal lattice Δ (black lines) with Powell–Sabin 6-split Δ^* (black and dotted lines)

Let $k \in \Delta$, then the interpolation problem

$$\left[\phi(l), \frac{\partial}{\partial x}\phi(l), \frac{\partial}{\partial y}\phi(l)\right] = \delta_{k,l}\left[\alpha, \beta, \gamma\right], \qquad l \in \Delta, \tag{1}$$

has a unique solution $\phi \in S_2^1(\Delta^*)$, see [10]. This allows to define a function vector $\boldsymbol{\phi} = [\phi_1, \phi_2, \phi_3]^T$ that generates a multiresolution analysis (MRA). Let each ϕ_i, $i = 1, 2, 3$, be the unique solution of (1) with

$$\alpha = \frac{1}{3},\ \beta = \frac{8\sqrt{3}}{9} - \delta_{i1}\frac{24\sqrt{3}}{9},\ \gamma = \left((-1)^{i-1} - \delta_{i1}\right)\frac{8}{3},$$

then the integer translates under Γ of the basis functions ϕ_i form a basis for $S_2^1(\Delta^*)$. Furthermore they form a partition of unity, see [3]. Define the 2×2 dilation matrix D given by

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

then we consider the refinement $\Delta_j := D^{-j}\Delta$, which can be obtained by midedge subdivision, and the corresponding PS 6-split $\Delta_j^* := D^{-j}\Delta^*$. This yields nested subspaces $V_j = S_2^1(\Delta_j^*) \subset L_2(\mathbb{R}^2)$, $j \in \mathbb{Z}$, such that

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}, \tag{2}$$

the closure of their union is $L_2(\mathbb{R}^2)$, and their intersection contains only the zero function. In general, the basis functions on all standard refinements Δ_j of Δ can be written as

$$\boldsymbol{\phi}_{j,k}(u) = \boldsymbol{\phi}(D^j(u-k)), \qquad k \in \Delta_j, \quad u \in \mathbb{R}^2,$$

and the set $\left\{2^j \boldsymbol{\phi}_{j,k} \mid k \in \Delta_j\right\}$ forms an L^2-stable basis of V_j, i.e., for $\mathbf{c} = \{\mathbf{c}_k\}_{k\in\Delta_j} \in l_2^{3\times 1}(\Delta_j)$, $\mathbf{c}_k := (c_{1,k}, c_{2,k}, c_{3,k})^T$,

$$\left\| \sum_{k\in\Delta_j} \mathbf{c}_k^T 2^j \boldsymbol{\phi}_{j,k} \right\|_{L_2(\mathbb{R}^2)} \sim \|\mathbf{c}\|_{l_2^{3\times 1}(\Delta_j)},$$

see [7, 8]. With $l_2^{3\times 1}(\Delta_j)$ we denote the Banach space of all sequences of 3×1 vectors $\mathbf{c}_k$ for which $\sqrt{\sum_{k\in\Delta_j} \|\mathbf{c}_k\|_2^2} < \infty$, and we always mean by $a \sim b$ that a can be bounded above and below by constant multiples of b.

Because of properties listed above we say that the sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^2)$ forms a MRA of multiplicity 3, and the function vector $\boldsymbol{\phi}$ is called scaling vector, see [9]. The nestedness (2) of the MRA implies that $\boldsymbol{\phi}$ needs to satisfy a matrix refinement equation of the form

$$\boldsymbol{\phi}(u) = \sum_{k\in\mathbb{Z}^2} \mathbf{A}_k \boldsymbol{\phi}(Du - \Gamma k), \quad u \in \mathbb{R}^2, \tag{3}$$

where $\mathbf{A}_k$ are 3×3 mask coefficient matrices. Moreover, $\mathbf{A}_{(-1,-1)}$ and $\mathbf{A}_{(0,-1)}$ are given by

$$\frac{1}{4}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$\mathbf{A}_{(-1,0)}$, $\mathbf{A}_{(0,0)}$ and $\mathbf{A}_{(1,0)}$ are given by

$$\frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \frac{1}{6}\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

and $\mathbf{A}_{(0,1)}$ and $\mathbf{A}_{(1,1)}$ are given by

$$\frac{1}{4}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix},$$

see e.g. [13].

2 Construction of a dual scaling vector

The main idea is to start with a hierarchical basis and apply a local correction process on the basis functions at each level in order to achieve certain regularity properties for the dual function vectors. This technique fits in the framework of both the lifting scheme [11] and the stable completion technique [1].

For a vertex $k \in \Delta_j$, we define $N_j(k)$ to be the neighbouring vertices in Δ_j of vertex k, and we set $\Lambda_j = \Delta_{j+1} \setminus \Delta_j$. Each scaling vector $\boldsymbol{\phi}_{j,k}$, $k \in \Delta_j$, is refinable in the sense that $\boldsymbol{\phi}_{j,k}$, $k \in \Delta_j$, can be written as linear combination of $\boldsymbol{\phi}_{j+1,l}$, $l \in \Delta_{j+1}$. In fact, using (3),

$$\boldsymbol{\phi}_{j,k} = \mathbf{A}_{(0,0)}\boldsymbol{\phi}_{j+1,k} + \sum_{l\in N_{j+1}(k)} \mathbf{A}_{\Gamma^{-1}D^{j+1}(l-k)}\boldsymbol{\phi}_{j+1,l}. \tag{4}$$

We define the hierarchical wavelets by

$$\boldsymbol{\psi}_{j,l}^h = \boldsymbol{\phi}_{j+1,l}, \qquad l \in \Lambda_j. \tag{5}$$

From (4) we get that each $\boldsymbol{\phi}_{j+1,l}$, $l \in \Delta_{j+1}$, can be obtained as a linear combination of the scaling vectors $\boldsymbol{\phi}_{j,k}$, $k \in \Delta_j$, and the wavelet vectors $\boldsymbol{\psi}_{j,l}^h$, $l \in \Lambda_j$, according to

$$\boldsymbol{\phi}_{j+1,l} = \begin{cases} \mathbf{A}_{(0,0)}^{-1}\Big[\boldsymbol{\phi}_{j,l} - \\ \quad \sum_{\lambda\in N_{j+1}(l)} \mathbf{A}_{\Gamma^{-1}D^{j+1}(\lambda-l)}\boldsymbol{\psi}_{j,\lambda}^h\Big], & l \in \Delta_j, \\ \boldsymbol{\psi}_{j,l}^h, & l \in \Lambda_j. \end{cases} \tag{6}$$

Equation (6) implies that the space W_j^h spanned by $\boldsymbol{\psi}_{j,l}^h$, $l \in \Lambda_j$, is a complement space of V_j into V_{j+1}, and we have

$$V_J = V_0 \oplus W_0^h \oplus \cdots \oplus W_{J-1}^h,$$

so that $\left\{\boldsymbol{\phi}_{0,k}, k \in \Delta_0\right\} \cup \left\{\boldsymbol{\psi}_{j,l}^h, l \in \Lambda_j, 0 \le j \le J\right\}$ is a basis of V_J. The hierarchical wavelet basis can be viewed as a biorthogonal wavelet basis, where the duals exist in $L_2(\mathbb{R}^2)$ in the distributional sense. We define the dual scaling vectors by

$$\tilde{\boldsymbol{\phi}}_{j,k}^h = \begin{bmatrix} \delta_k - 2^{-j-1}\delta_k \frac{\partial}{\partial x} \\ \delta_k + 2^{-j-2}\delta_k \frac{\partial}{\partial x} - \sqrt{3}\, 2^{-j-2}\delta_k \frac{\partial}{\partial y} \\ \delta_k + 2^{-j-2}\delta_k \frac{\partial}{\partial x} + \sqrt{3}\, 2^{-j-2}\delta_k \frac{\partial}{\partial y} \end{bmatrix}, \tag{7}$$

with δ_k the Dirac distribution at vertex k, and the dual wavelets by

$$\begin{aligned} \widetilde{\boldsymbol{\psi}}^h_{j,l} &= \widetilde{\boldsymbol{\phi}}^h_{j+1,l} - \mathbf{A}^T_{\Gamma^{-1}D^{j+1}(l-k_1)}\mathbf{A}^{-1}_{(0,0)}\widetilde{\boldsymbol{\phi}}^h_{j+1,k_1} \\ &\quad -\mathbf{A}^T_{\Gamma^{-1}D^{j+1}(l-k_2)}\mathbf{A}^{-1}_{(0,0)}\widetilde{\boldsymbol{\phi}}^h_{j+1,k_2}, \end{aligned} \tag{8}$$

with $k_1, k_2 \in \Delta_j$ such that $N_{j+1}(k_1) \cap N_{j+1}(k_2) = \{l\}$. Note that $\widetilde{\boldsymbol{\phi}}^h_{j,k}$ satisfies the refinement equation

$$\widetilde{\boldsymbol{\phi}}^h_{j,k} = \mathbf{A}^{-1}_{(0,0)}\widetilde{\boldsymbol{\phi}}^h_{j+1,k}, \qquad k \in \Delta_j. \tag{9}$$

The sets $\left\{\boldsymbol{\phi}_{j,k}, \boldsymbol{\psi}^h_{j,l}\right\}$ and $\left\{\widetilde{\boldsymbol{\phi}}^h_{j,k}, \widetilde{\boldsymbol{\psi}}^h_{j,l}\right\}$ are biorthogonal, i.e.

$$\begin{cases} \langle\widetilde{\boldsymbol{\phi}}^h_{j,k'}, \left(\boldsymbol{\phi}_{j,k}\right)^T\rangle &= \delta_{k,k'}I_3, \qquad k,k' \in \Delta_j, \\ \langle\widetilde{\boldsymbol{\phi}}^h_{j,k'}, \left(\boldsymbol{\psi}^h_{j,l}\right)^T\rangle &= 0, \qquad l \in \Lambda_j, k' \in \Delta_j, \\ \langle\widetilde{\boldsymbol{\psi}}^h_{j,l'}, \left(\boldsymbol{\phi}_{j,k}\right)^T\rangle &= 0, \qquad k \in \Delta_j, l' \in \Lambda_j \\ \langle\widetilde{\boldsymbol{\psi}}^h_{j,l'}, \left(\boldsymbol{\psi}^h_{j,l}\right)^T\rangle &= \delta_{l,l'}I_3, \qquad l,l' \in \Lambda_j, \end{cases}$$

with $\langle\cdot,\cdot\rangle$ the inner product in $L_2(\mathbb{R}^2)$. However, from (7) we immediately find that the interpolation operator Q^h_j,

$$Q^h_j f = \sum_{k\in\Delta_j}\left(\langle\widetilde{\boldsymbol{\phi}}^h_{j,k}, f\rangle\right)^T \boldsymbol{\phi}_{j,k},$$

can only be applied to C^1 continuous functions. Therefore the hierarchical wavelet basis is restricted to the decomposition of Sobolev spaces H^s with $s > 2$, since for $s \leq 2$ these spaces are not embedded in C^1, see also [7].

In order to enlarge the range of stability to Sobolev spaces H^s, $s \leq 2$, we shall use the lifting scheme [11]. Define $\mathbf{\Phi}_j$ as the column vector containing all scaling vectors $\boldsymbol{\phi}_{j,k}$ for all $k \in \Delta_j$, and define $\mathbf{\Psi}^h_j$ likewise. Equations (4), (5), (8) and (9) give us a set of biorthogonal filter operators $\left\{\mathbf{A}_j, \mathbf{B}_j, \widetilde{\mathbf{A}}_j, \widetilde{\mathbf{B}}_j\right\}$ such that

$$\begin{aligned} \begin{bmatrix}\mathbf{\Phi}_j & \mathbf{\Psi}^h_j\end{bmatrix}^T &= \begin{bmatrix}\mathbf{A}^T_j & \mathbf{B}^T_j\end{bmatrix}^T \mathbf{\Phi}_{j+1} \\ \mathbf{\Phi}_{j+1} &= \begin{bmatrix}\widetilde{\mathbf{A}}^T_j & \widetilde{\mathbf{B}}^T_j\end{bmatrix}\begin{bmatrix}\mathbf{\Phi}_j & \mathbf{\Psi}^h_j\end{bmatrix}^T \end{aligned}$$

and

$$\begin{bmatrix}\mathbf{A}_j \\ \mathbf{B}_j\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{A}}^T_j & \widetilde{\mathbf{B}}^T_j\end{bmatrix} = I.$$

The lifting scheme generates new wavelet functions $\mathbf{\Psi}_j$ by projecting the hierarchical wavelet functions $\mathbf{\Psi}^h_j$

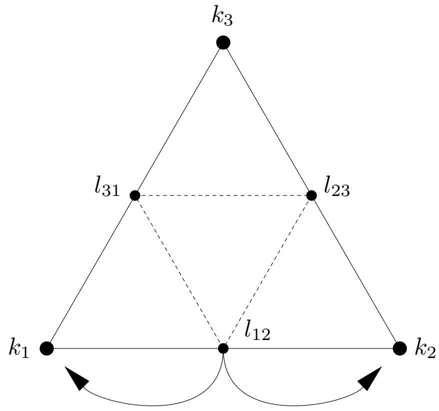

Figure 2: The update stencil

into the desired complement space W_j along V_j,

$$\mathbf{\Psi}_j = \mathbf{\Psi}^h_j - \mathbf{C}_j\mathbf{\Phi}_j.$$

This projection is not necessarily orthogonal. For each wavelet function there is a corresponding row in the lifting matrix $\mathbf{C}_j$. The possibly nonzero entries in this row together will be called the update stencil for that wavelet function. Hence, the lifting scheme provides us with a new set of biorthogonal filter operators satisfying

$$\begin{bmatrix}\mathbf{A}_j \\ \mathbf{B}_j - \mathbf{C}_j\mathbf{A}_j\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{A}}^T_j + \widetilde{\mathbf{B}}^T_j\mathbf{C}_j & \widetilde{\mathbf{B}}^T_j\end{bmatrix} = I.$$

In order to understand the stability of this new basis with respect to Sobolev spaces, we have to investigate the regularity of the corresponding dual functions $\widetilde{\mathbf{\Phi}}_j$ satisfying

$$\widetilde{\mathbf{\Phi}}_j = \widetilde{\mathbf{A}}_j\widetilde{\mathbf{\Phi}}_{j+1} + \mathbf{C}^T_j\widetilde{\mathbf{B}}_j\widetilde{\mathbf{\Phi}}_{j+1}. \tag{10}$$

Therefore we build the lifting matrix $\mathbf{C}_j$ in such a way that the dual functions (10) are well defined and that they are more regular than the initial Dirac distributions (7). We fix the update stencil in advance in order to have local support for the wavelet functions, see Figure 2. Each wavelet vector $\boldsymbol{\psi}_{j,l}$ corresponding to a new vertex $l \in \Lambda_j$ is updated by the scaling vectors $\boldsymbol{\phi}_{j,k_1}, \boldsymbol{\phi}_{j,k_2}$ with $k_1, k_2 \in \Delta_j$ such that $N_{j+1}(k_1) \cap N_{j+1}(k_2) = \{l\}$. Because there are three scaling functions associated with each scaling vector, the width of the update stencil for a wavelet function is six. We want to orthogonalise the wavelets to their predefined set of scaling functions, but we also want one vanishing moment. This leads to an overdetermined system because we do not have sufficient degrees of freedom. The lifting matrix $\mathbf{C}_j$ is constructed as the least squares solution to this overdetermined system but we demand that the vanishing moment condition is fulfilled. Denote the vertices $k_1, k_2, k_3 \in \Delta_j$ and $l_{12}, l_{23}, l_{31} \in \Lambda_j$ as in Figure 2. We find that

$$\boldsymbol{\psi}_{j,l_{12}} = \boldsymbol{\psi}^h_{j,l_{12}} - \left[\begin{array}{ccc} \frac{-10815353}{37042932} & \frac{489004135}{1926232464} & \frac{489004135}{1926232464} \\ \frac{-20302895}{148171728} & \frac{734318841}{4494542416} & \frac{179538083}{3370906812} \\ \frac{-20302895}{148171728} & \frac{179538083}{3370906812} & \frac{734318841}{4494542416} \end{array} \right.$$

$$\left. \begin{array}{ccc} \frac{52104049}{481558116} & \frac{-71233997}{1926232464} & \frac{-71233997}{1926232464} \\ \frac{681707773}{1926232464} & \frac{-164964803}{4494542416} & \frac{-247462325}{1685453406} \\ \frac{681707773}{1926232464} & \frac{-247462325}{1685453406} & \frac{-164964803}{4494542416} \end{array} \right] \left[\begin{array}{c} \boldsymbol{\phi}_{j,k_1} \\ \boldsymbol{\phi}_{j,k_2} \end{array} \right]$$

$$\boldsymbol{\psi}_{j,l_{23}} = \boldsymbol{\psi}^h_{j,l_{23}} - \left[\begin{array}{ccc} \frac{734318841}{4494542416} & \frac{-20302895}{148171728} & \frac{179538083}{3370906812} \\ \frac{489004135}{1926232464} & \frac{-10815353}{37042932} & \frac{489004135}{1926232464} \\ \frac{179538083}{3370906812} & \frac{-20302895}{148171728} & \frac{734318841}{4494542416} \end{array} \right.$$

$$\left. \begin{array}{ccc} \frac{-164964803}{4494542416} & \frac{681707773}{1926232464} & \frac{-247462325}{1685453406} \\ \frac{-71233997}{1926232464} & \frac{52104049}{481558116} & \frac{-71233997}{1926232464} \\ \frac{-247462325}{1685453406} & \frac{681707773}{1926232464} & \frac{-164964803}{4494542416} \end{array} \right] \left[\begin{array}{c} \boldsymbol{\phi}_{j,k_2} \\ \boldsymbol{\phi}_{j,k_3} \end{array} \right]$$

$$\boldsymbol{\psi}_{j,l_{31}} = \boldsymbol{\psi}^h_{j,l_{31}} - \left[\begin{array}{ccc} \frac{734318841}{4494542416} & \frac{179538083}{3370906812} & \frac{-20302895}{148171728} \\ \frac{179538083}{3370906812} & \frac{734318841}{4494542416} & \frac{-20302895}{148171728} \\ \frac{489004135}{1926232464} & \frac{489004135}{1926232464} & \frac{-10815353}{37042932} \end{array} \right.$$

$$\left. \begin{array}{ccc} \frac{-164964803}{4494542416} & \frac{-247462325}{1685453406} & \frac{681707773}{1926232464} \\ \frac{-247462325}{1685453406} & \frac{-164964803}{4494542416} & \frac{681707773}{1926232464} \\ \frac{-71233997}{1926232464} & \frac{-71233997}{1926232464} & \frac{52104049}{481558116} \end{array} \right] \left[\begin{array}{c} \boldsymbol{\phi}_{j,k_3} \\ \boldsymbol{\phi}_{j,k_1} \end{array} \right]$$

See also [12] for a similar construction on arbitrary polygonal domains. The dual scaling vectors $\widetilde{\boldsymbol{\phi}}_{j,k}$ satisfy a matrix refinement relation similar to (4), and using (10) we find that

$$\begin{aligned} \widetilde{\boldsymbol{\phi}}_{j,k} &= \widetilde{\mathbf{A}}_{(0,0)} \widetilde{\boldsymbol{\phi}}_{j+1,k} \\ &+ \sum_{l \in N_{j+1}(k)} \widetilde{\mathbf{A}}_{\Gamma^{-1} D^{j+1}(l-k)} \widetilde{\boldsymbol{\phi}}_{j+1,l} \\ &+ \sum_{l \in N_j(k)} \widetilde{\mathbf{A}}_{\Gamma^{-1} D^{j+1}(l-k)} \widetilde{\boldsymbol{\phi}}_{j+1,l}, \end{aligned} \qquad (11)$$

where $\widetilde{\mathbf{A}}_{(-2,-2)}$ and $\widetilde{\mathbf{A}}_{(0,-2)}$ are given by

$$\left[\begin{array}{ccc} \frac{-37}{1344} & \frac{37}{1344} & \frac{-417301265}{2568309952} \\ \frac{37}{1344} & \frac{-37}{1344} & \frac{-417301265}{2568309952} \\ 0 & 0 & \frac{20302895}{98781152} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{1494625759}{53934508992} & \frac{-212816425}{7704929856} & \frac{1488078047}{17978169664} \\ \frac{-366180055}{1926232464} & \frac{366180055}{3852464928} & \frac{-366180055}{1926232464} \\ \frac{1488078047}{17978169664} & \frac{-212816425}{7704929856} & \frac{1494625759}{53934508992} \end{array} \right],$$

$\widetilde{\mathbf{A}}_{(-1,-1)}$ and $\widetilde{\mathbf{A}}_{(0,-1)}$ are given by

$$\left[\begin{array}{ccc} \frac{734318841}{4494542416} & \frac{179538083}{3370906812} & \frac{489004135}{1926232464} \\ \frac{179538083}{3370906812} & \frac{734318841}{4494542416} & \frac{489004135}{1926232464} \\ \frac{-20302895}{148171728} & \frac{-20302895}{148171728} & \frac{-10815353}{37042932} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{-164964803}{4494542416} & \frac{-71233997}{1926232464} & \frac{-247462325}{1685453406} \\ \frac{681707773}{1926232464} & \frac{52104049}{481558116} & \frac{681707773}{1926232464} \\ \frac{-247462325}{1685453406} & \frac{-71233997}{1926232464} & \frac{-164964803}{4494542416} \end{array} \right],$$

$\widetilde{\mathbf{A}}_{(-2,0)}$, $\widetilde{\mathbf{A}}_{(-1,0)}$, $\widetilde{\mathbf{A}}_{(0,0)}$, $\widetilde{\mathbf{A}}_{(1,0)}$ and $\widetilde{\mathbf{A}}_{(2,0)}$ are given by

$$\left[\begin{array}{ccc} \frac{366180055}{3852464928} & \frac{-366180055}{1926232464} & \frac{-366180055}{1926232464} \\ \frac{-212816425}{7704929856} & \frac{1494625759}{53934508992} & \frac{1488078047}{17978169664} \\ \frac{-212816425}{7704929856} & \frac{1488078047}{17978169664} & \frac{1494625759}{53934508992} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{52104049}{481558116} & \frac{681707773}{1926232464} & \frac{681707773}{1926232464} \\ \frac{-71233997}{1926232464} & \frac{-164964803}{4494542416} & \frac{-247462325}{1685453406} \\ \frac{-71233997}{1926232464} & \frac{-247462325}{1685453406} & \frac{-164964803}{4494542416} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{417002791}{842726703} & \frac{-40250807}{13483627248} & \frac{-40250807}{13483627248} \\ \frac{-40250807}{13483627248} & \frac{417002791}{842726703} & \frac{-40250807}{13483627248} \\ \frac{-40250807}{13483627248} & \frac{-40250807}{13483627248} & \frac{417002791}{842726703} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{-10815353}{37042932} & \frac{-20302895}{148171728} & \frac{-20302895}{148171728} \\ \frac{489004135}{1926232464} & \frac{734318841}{4494542416} & \frac{179538083}{3370906812} \\ \frac{489004135}{1926232464} & \frac{179538083}{3370906812} & \frac{734318841}{4494542416} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{20302895}{98781152} & 0 & 0 \\ \frac{-417301265}{2568309952} & \frac{-37}{1344} & \frac{37}{1344} \\ \frac{-417301265}{2568309952} & \frac{37}{1344} & \frac{-37}{1344} \end{array} \right],$$

$\widetilde{\mathbf{A}}_{(0,1)}$ and $\widetilde{\mathbf{A}}_{(1,1)}$ are given by

$$\left[\begin{array}{ccc} \frac{734318841}{4494542416} & \frac{489004135}{1926232464} & \frac{179538083}{3370906812} \\ \frac{-20302895}{148171728} & \frac{-10815353}{37042932} & \frac{-20302895}{148171728} \\ \frac{179538083}{3370906812} & \frac{489004135}{1926232464} & \frac{734318841}{4494542416} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{-164964803}{4494542416} & \frac{-247462325}{1685453406} & \frac{-71233997}{1926232464} \\ \frac{-247462325}{1685453406} & \frac{-164964803}{4494542416} & \frac{-71233997}{1926232464} \\ \frac{681707773}{1926232464} & \frac{681707773}{1926232464} & \frac{52104049}{481558116} \end{array} \right],$$

and $\widetilde{\mathbf{A}}_{(0,2)}$ and $\widetilde{\mathbf{A}}_{(2,2)}$ are given by

$$\left[\begin{array}{ccc} \frac{-37}{1344} & \frac{-417301265}{2568309952} & \frac{37}{1344} \\ 0 & \frac{20302895}{98781152} & 0 \\ \frac{37}{1344} & \frac{-417301265}{2568309952} & \frac{-37}{1344} \end{array} \right],$$

$$\left[\begin{array}{ccc} \frac{1494625759}{53934508992} & \frac{1488078047}{17978169664} & \frac{-212816425}{7704929856} \\ \frac{1488078047}{17978169664} & \frac{1494625759}{53934508992} & \frac{-212816425}{7704929856} \\ \frac{-366180055}{1926232464} & \frac{-366180055}{1926232464} & \frac{366180055}{3852464928} \end{array} \right].$$

Equation (11) allows to look for a solution of the form

$$\widetilde{\boldsymbol{\phi}}_{j,k}(u) = 4^j \widetilde{\boldsymbol{\phi}}(D^j(u-k)),\ k \in \Delta_j,\ u \in \mathbb{R}^2.$$

Hence, the vector $\widetilde{\boldsymbol{\phi}}$ is the solution of the refinement equation

$$\widetilde{\boldsymbol{\phi}}(u) = 4 \sum_{k \in \mathbb{Z}^2} \widetilde{\mathbf{A}}_k \widetilde{\boldsymbol{\phi}}(Du - \Gamma k), \quad u \in \mathbb{R}^2, \qquad (12)$$

and the regularity of the duals $\widetilde{\boldsymbol{\phi}}_{j,k}$ equals the regularity of the distributional solution to (12).

3 Existence, uniqueness and regularity of $\widetilde{\phi}$

Taking the Fourier transform of both sides of (12), we obtain

$$\widehat{\widetilde{\phi}}(\omega) = \widetilde{\mathbf{P}}(D^{-T}\omega)\widehat{\widetilde{\phi}}(D^{-T}\omega), \quad \omega \in \mathbb{R}^2,$$

and

$$\widetilde{\mathbf{P}}(\omega) := \sum_{k \in \mathbb{Z}^2} \widetilde{\mathbf{A}}_k e^{-i(\Gamma k \cdot \omega)}, \quad \omega \in \mathbb{R}^2,$$

is the symbol associated with (12). It is easily checked that $\widetilde{\mathbf{P}}(0)$ satisfies Condition E, i.e., 1 is a simple eigenvalue of $\widetilde{\mathbf{P}}(0)$ and all other eigenvalues of $\widetilde{\mathbf{P}}(0)$ lie inside the open unit disk. Let $\widetilde{\mathbf{r}}$ be the normalized right eigenvector of $\widetilde{\mathbf{P}}(0)$ associated with eigenvalue 1. It is well-known that if Condition E holds then there exists a unique compactly supported distributional solution vector $\widetilde{\phi}$ satisfying (12) with $\widehat{\widetilde{\phi}}(0) = \widetilde{\mathbf{r}}$, see [9]. In particular, we have

$$\widehat{\widetilde{\phi}}(\omega) = \lim_{L \to \infty} \prod_{j=1}^{L} \widetilde{\mathbf{P}}\left(\left(D^{-T}\right)^j \omega\right) \widetilde{\mathbf{r}}.$$

The regularity or smoothness of $\widetilde{\phi}$ is measured by the critical exponent

$$s(\widetilde{\phi}) := \sup\left\{ s : \widetilde{\phi}_i \in H^s(\mathbb{R}^2) \text{ for all } i = 1, 2, 3 \right\},$$

where $H^s(\mathbb{R}^2)$ denotes the Sobolev space $\{f \in L_2(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\widehat{f}(\xi)|^2 (1 + |\xi|^s)^2 d\xi < \infty\}$. Theorem 5.3 in [4] gives an estimate for the critical Sobolev exponent of a vector of compactly supported functions in $L_2(\mathbb{R}^2)$ satisfying a refinement equation. We use this theorem to prove the following proposition.

Proposition 3.1. *Let $\widetilde{\phi}$ be the unique 3×1 solution vector of compactly supported distributions satisfying (12). Then $\widetilde{\phi}$ is in the Sobolev space $H^s(\mathbb{R}^2)$ for any*

$$s < -0.431898.$$

Proof. Let $\mathbf{b}$ be given by

$$\mathbf{b}(l) = 4 \sum_{k \in \mathbb{Z}^2} \widetilde{\mathbf{A}}_k \otimes \widetilde{\mathbf{A}}_{k+l},$$

$l \in \mathbb{Z}^2$, where $\otimes$ denotes the (right) Kronecker product. Then $\mathbf{b}$ is supported in $[-4, 4]^2$. Define K as the set $\mathbb{Z}^2 \cap \left(\sum_{j=1}^{\infty} D^{-j}(\operatorname{supp} \mathbf{b})\right)$. Then K equals $[-4, 4]^2$. Let $\mathbf{B}$ be the 729×729 matrix $(\mathbf{b}\,(Dk - l))_{k,l \in K}$. Theorem 5.3 in [4] asserts that $s(\widetilde{\phi}) \geq -\log_4 \rho$, where $\rho = \max\{|\nu| : \nu \in \operatorname{spec}(\mathbf{B})\}$. The largest eigenvalue of $\mathbf{B}$ is given by 1.819820559284, hence we obtain that $s(\widetilde{\phi}) \geq -0.431898$. □

It is well-known that the scaling vector ϕ satisfying (3) is in the Sobolev space $H^s(\mathbb{R}^2)$ for any $s < 5/2$. One can, for instance, use a similar proof as in Proposition 3.1 and use Theorem 5.3 in [4], or one can make use of techniques with Jackson and Bernstein inequalities, see [7].

We are interested in determining the exact range of Sobolev exponents s for which the wavelet basis forms a Riesz basis for $H^s(\mathbb{R}^2)$. From important work by Dahmen [2] and Lorentz and Oswald [6] it turns out that the range of such s is determined by the Sobolev regularity $s(\phi)$ of the scaling vector ϕ and the Sobolev regularity $s(\widetilde{\phi})$ of the dual scaling vector $\widetilde{\phi}$. Dahmen showed that the wavelet system is a Riesz basis for $H^s(\mathbb{R}^2)$ for all s with $-s(\widetilde{\phi}) < s < s(\phi)$ and that this interval is sharp, provided that ϕ and $\widetilde{\phi}$ have compact support and that they are in $L_2(\mathbb{R}^2)$. Lorentz and Oswald extended these results to non-compact $\widetilde{\phi} \notin L_2(\mathbb{R}^2)$. The following theorem is our main result and is a direct consequence of Proposition 3.1 and the work in [2] and [6].

Theorem 3.2. *One has for any $f \in H^s(\mathbb{R}^2)$, $0.431898 < s < 5/2$, the norm equivalence*

$$\left\| \langle \widetilde{\phi}_{j_0,k}, f \rangle \right\|^2_{l_2^{3\times 1}(\Delta_{j_0})} + \sum_{j=j_0}^{\infty} 2^{2j(s-1)} \left\| \langle \widetilde{\psi}_{j,l}, f \rangle \right\|^2_{l_2^{3\times 1}(\Lambda_j)} \sim \|f\|^2_{H^s(\mathbb{R}^2)}. \tag{13}$$

We close this section with some graphs of the scaling function, the dual scaling function and the wavelet function, see Figures 3 to 5. The graph of the dual scaling function was generated using software by Q. Jiang and P. Oswald, see [5].

4 Applications

Several approaches to solving elliptic problems numerically are based on hierarchical Riesz bases in Sobolev spaces. Theorem 3.2 suggests that the above constructed multiwavelet basis is suitable for solving second and fourth order elliptic equations since the norm equivalence (13) includes the values $s = 1$ and $s = 2$.

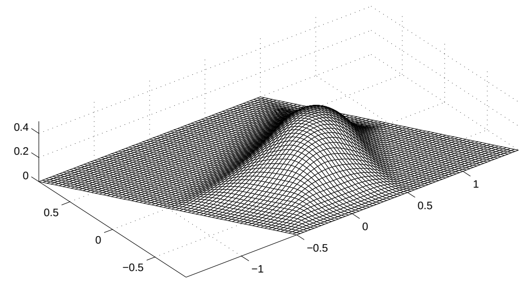

Figure 3: Scaling function

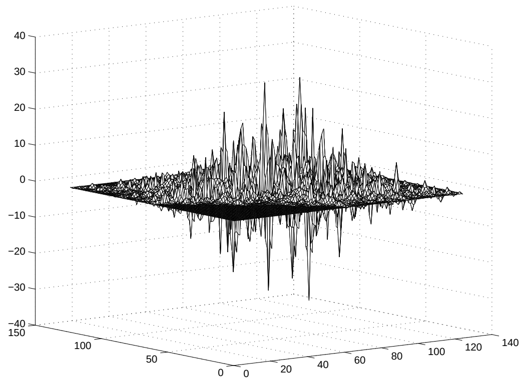

Figure 4: Dual scaling function

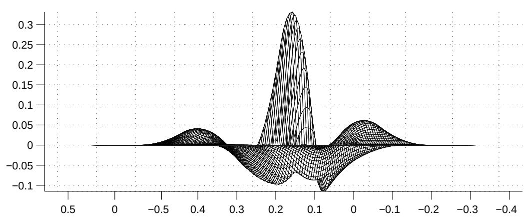

Figure 5: Wavelet function

	Lena	Barbara	Fingerprint
Nr. of coeff.	63145	69893	69510
Percentage	8.00%	8.85%	8.80%

Table 1: Number of coefficients kept in the compressed images

However, with such applications in mind we need to adapt the multiwavelet basis to polygonal domains in $\mathbb{R}^2$ instead of the whole plane. We will deal with this problem in a forthcoming paper.

To demonstrate the approximation power of the wavelet basis we show their performance on the compression of images. We can interpret a grayscale image as a surface, where the value of each pixel represents its height. We shall use 8-bit grayscale images, which means the pixel values range from 0 (black) to 255 (white), and the image array consists of 513×513 pixel values. First we interpolate the image by solving the interpolation problem (1) for each pixel. Derivative information of the image surface can be estimated by applying the Sobel operator to the image. Then we apply the wavelet transform and we compress the image by replacing all wavelet coefficients which have an absolute value smaller than a threshold value by zero. The compression results are depicted in Figures 6 to 8. Before the compression step we have 789507 coefficients. The number of coefficients after the compression step can be found in Table 1.

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(a) Original

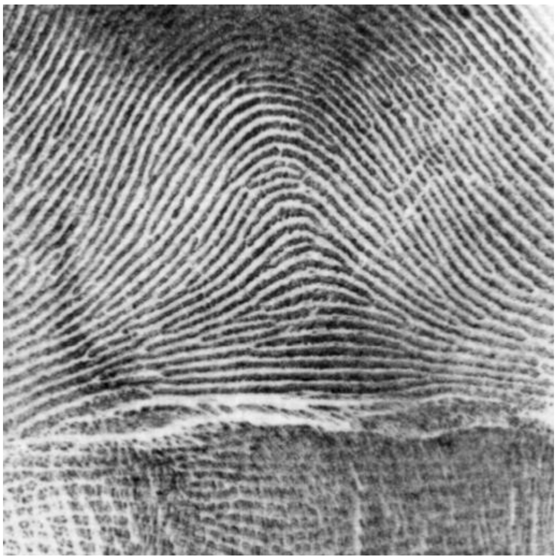
(b) Compressed, 12.5:1

Figure 6: Lena

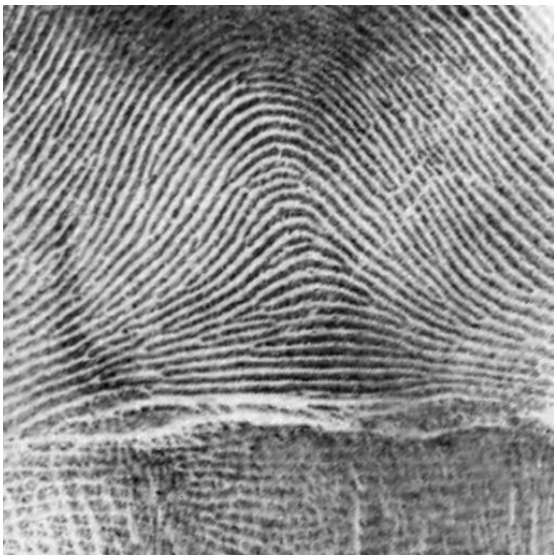
(a) Original

(b) Compressed, 11:1

Figure 7: Barbara



(a) Original



(b) Compressed, 11:1

Figure 8: Fingerprint

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