ORTHOGONALITY AND BOUNDARY INTERPOLATION

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Abstract

Let $\{\alpha_n : n = 1, 2, \dots\}$ be a sequence of (not necessarily distinct) points on the unit circle. Set $\omega_0(z) \equiv 1$, $\omega_n(z) = (z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n)$ for $n = 1, 2, \dots$, and let $\mathcal{L}$ denote the linear space spanned by the functions $\{1/\omega_n : n = 0, 1, 2, \dots\}$. Let M be a quasi-definite linear functional on $\mathcal{L} \cdot \mathcal{L}$, and define the inner product $\langle, \rangle$ by $\langle f, g \rangle = M(fg_*)$, where $g_*(z) = \overline{g(1/\bar{z})}$. (In particular M may be a positive definite functional given by $M(f) = \int_{-\pi}^{\pi} f(e^{it}) d\mu(t)$, where μ is a measure such that all functions in $\mathcal{L} \cdot \mathcal{L}$ are μ -integrable). Let $\{\varphi_n\}$ be an orthogonal system obtained from the base $\{1/\omega_n\}$, and define the associated functions $\{\psi_n\}$ by $\psi_n(z) = M(\frac{t+z}{t-z}[\varphi_n(t) - \varphi_n(z)])$. Interpolation of ψ_n/φ_n to $M(\frac{t+z}{t-z})$ (or to the formal series $\mu_0 + 2 \sum_{m=1}^{\infty} \mu_m z \omega_{m-1}(z)$, $\mu_m = M(1/\omega_m)$) at interpolation points α_k is discussed.

1 Introduction

Let $\{\alpha_n\}$ be a sequence of (not necessarily distinct) points on the unit circle $T = \{z \in C : |z| = 1\}$. We define the polynomials ω_n by

$$\omega_0 = 1, \quad \omega_n(z) = \prod_{m=1}^n (z - \alpha_m), \quad n = 1, 2, \dots \quad (1.1)$$

The linear space spanned by the functions $\{\frac{1}{\omega_0}, \frac{1}{\omega_1}, \dots, \frac{1}{\omega_n}\}$ shall be denoted by $\mathcal{L}_n$, i.e.

$$\mathcal{L}_n = \text{Span} \left\{ \frac{1}{\omega_0}, \frac{1}{\omega_1}, \dots, \frac{1}{\omega_n} \right\}. \quad (1.2)$$

Further we define

$$\mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n. \quad (1.3)$$

The functions in $\mathcal{L}_n$ are exactly the function that may be written in the form

$$R(z) = \frac{p(z)}{\omega_n(z)}, \quad p \in \Pi_n, \quad (1.4)$$

where Π_n denotes the space of polynomials of degree at most n . (This follows from the basic facts about partial fraction decomposition.)

Let M be a given linear functional defined at least on $\mathcal{L} \cdot \mathcal{L}$. (Note that M is then defined on $\mathcal{L}$, since $\mathcal{L} = \mathcal{L} \cdot 1 \subset \mathcal{L} \cdot \mathcal{L}$.) The functional M may in particular be defined by

$$M[R] = \int_{-\pi}^{\pi} R(e^{i\theta}) d\mu(\theta), \quad (1.5)$$

where μ is a finite Borel measure on $[-\pi, \pi]$ with the property that all the integrals $\int_{-\pi}^{\pi} [\omega_m(e^{i\theta})\omega_n(e^{i\theta})]^{-1} d\mu(\theta)$, $m, n = 0, 1, 2, \dots$, exist.

For every function f on $\hat{C}$ we define its substar transform f_* by

$$f_*(z) = \overline{f(1/\bar{z})}. \quad (1.6)$$

We note that if $R \in \mathcal{L}_n$, then also $R_* \in \mathcal{L}_n$, since

$$\left[\frac{1}{\omega_m(z)} \right]_* = (-1)^m \alpha_1 \dots \alpha_m \frac{z^m}{\omega_m(z)}. \quad (1.7)$$

We define the Hermitian form $\langle , \rangle$ on $\mathcal{L} \times \mathcal{L}$ by

$$\langle P, Q \rangle = M[P \cdot Q_*]. \quad (1.8)$$

We shall assume that this form is quasi-definite, i.e.

$$\langle P, P \rangle \neq 0, \quad \text{for } P \in \mathcal{L}, P \neq 0. \quad (1.9)$$

This is in particular the case for the form associated by (1.5), which is positive definite.

Let $\{\varphi_n\}$ be the (essentially unique) orthonormal system associated with the sequence $\{\frac{1}{\omega_n}\}$. We may write

$$\varphi_n(z) = \frac{p_n(z)}{\omega_n(z)}, \quad p_n \in \Pi_n. \quad (1.10)$$

The functions of the second kind $\{\psi_n\}$ (associated with $\{\varphi_n\}$) are defined by

$$\psi_0 = 1, \quad \psi_n(z) = M\left[\frac{t+z}{t-z} [\varphi_n(t) - \varphi_n(z)]\right], \quad n = 1, 2, \dots \quad (1.11)$$

It is easily seen that the expression in the brackets $[]$ belongs to $\mathcal{L}_n$ both as a function of z and as a function of t , and M is here meant to operate on its argument considered as a function of t . The function ψ_n then belongs to $\mathcal{L}_n$, and it may be written as

$$\psi_n(z) = \frac{q_n(z)}{\omega_n(z)}, \quad q_n \in \Pi_n. \quad (1.12)$$

Let us now assume that the functional M is defined also for the functions $\eta_z^{(k)}(t) = \frac{z^j t^k}{t-z}$ for $j, k = 0, 1, z \in \hat{C} - T$. (By M being defined for the function $f(z, t)$ for $z = \infty$ we mean that M is defined for the function $f(\zeta^{-1}, t)$ for $\zeta = 0$). Then M is defined on the extended space $\tilde{\mathcal{L}} \cdot \tilde{\mathcal{L}}$, where

$$\tilde{\mathcal{L}} = \text{Span}\left\{\frac{1}{\omega_n(t)}, \frac{z^j t^k}{t-z} : n = 0, 2, 3, \dots, j, k = 0, 1, z \in \hat{C} - T\right\}. \quad (1.13)$$

We note that M is then in particular defined for t and t^2 .

We may then define the Riesz-Herglotz transform Ω_M of M by

$$\Omega_M(z) = M\left[\frac{t+z}{t-z}\right]. \quad (1.14)$$

(Here again M operates on its argument considered as a function of t .) The functional M defined by (1.5) is defined on $\tilde{\mathcal{L}} \cdot \tilde{\mathcal{L}}$. For the transform Ω_M of this functional we shall also write Ω_μ , thus

$$\Omega_\mu(z) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta). \quad (1.15)$$

The function $g_t(z) = \frac{t+z}{t-z}$ has formal Newton series expansions

$$\frac{t+z}{t-z} \sim 1 + 2 \sum_{k=1}^{\infty} \frac{z\omega_{k-1}(z)}{\omega_k(t)} \quad (1.16)$$

and

$$\frac{t+z}{t-z} \sim -1 + 2 \sum_{k=1}^{\infty} \frac{\alpha_k t^k \omega_{k-1}(z)}{z^k \omega_k(t)}. \quad (1.17)$$

(Cf. Section 2. For Newton series expansions, see also [12].) By operating formally by M on (1.16) and (1.17) we get the Newton series expansions

$$\Omega_M(z) \sim \mu_0 + 2 \sum_{k=1}^{\infty} \mu_k z \omega_{k-1}(z) \quad (1.18)$$

and

$$\Omega_M(z) \sim \nu_0 + 2 \sum_{k=1}^{\infty} \nu_k \frac{\omega_{k-1}(z)}{z^k}, \quad (1.19)$$

where the moments $\{\mu_n\}$, $\{\nu_n\}$ are defined by

$$\mu_n = M\left[\frac{1}{\omega_n(t)}\right], \quad n = 0, 1, 2, \dots \quad (1.20)$$

and

$$\nu_0 = -\mu_0, \quad \nu_n = \alpha_n M\left[\frac{t^n}{\omega_n(t)}\right], \quad n = 1, 2, \dots \quad (1.21)$$

We define the polynomials P_n and Q_n by

$$P_n(z) = \mu_0 + 2 \sum_{k=1}^n \mu_k z \omega_{k-1}(z), \quad (1.22)$$

$$Q_n(z) = \nu_0 + 2 \sum_{k=1}^n \nu_k \frac{\omega_{k-1}(z)}{z^k}. \quad (1.23)$$

For basic information concerning orthogonal functions associated with spaces $\mathcal{L}_n$ as defined above, see [8], cf. also [7,9].

We shall in this paper discuss some interpolation properties connecting the rational functions $\{\frac{\psi_n(z)}{\varphi_n(z)}\}$, the Riesz-Herglotz transform $\Omega_M(z)$ and the Newton series (1.18)-(1.19).

It will be pointed out that $\{\frac{\psi_n(z)}{\varphi_n(z)}\}$ is an (n, n) multipoint Padé approximant for $\Omega_M(z)$ at the interpolation table $\{0, \infty, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \dots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n\}$. - Related connections where all the interpolation points are in the interior of the unit circle are discussed e.g. in [4,5,6,7,16] and in references found there.

A special case of the situation treated in this paper is when the sequence $\{\alpha_n\}$ consists of a finite number of points cyclically repeated. The closely related (through the Cayley transform) cyclic situation when the points $\{\alpha_n\}$ lie on the real axis has been investigated e.g. in [14,15,16] and in the references found there. When in particular two points are cyclically repeated (for convenience ∞ and 0), the orthogonal functions are orthogonal Laurent polynomials. (See e.g. [13,17]. Cf. also [3].) When all the points $\{\alpha_n\}$ coalesce to one point (for convenience ∞), the orthogonal functions are orthogonal polynomials. (See e.g. [1,2,10].)

2 Interpolation results

In addition to the given interpolation points $\{\alpha_n\}$ on T , the point at the origin and at infinity will play a role in our discussion. We shall use the notation

$$\alpha_0 = 0, \quad \alpha_\infty = \infty. \quad (2.1)$$

Let A_n denote the interpolation table $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, and $\tilde{A}_n$ the table $\{\alpha_\infty, \alpha_1, \dots, \alpha_n\}$.

We note that the space $\mathcal{L}_n$ is independent of the order of the points $\{\alpha_1, \dots, \alpha_n\}$ and the space $\mathcal{L}$ is independent of the order of the points $\{\alpha_n : n = 1, 2, \dots\}$. We also note that for each n the functions $\{\varphi_0, \varphi_1, \dots, \varphi_n\}$ and $\{\psi_0, \psi_1, \dots, \psi_n\}$ are determined solely by the

table A_n (or $\tilde{A}_n$) and the values of the functional M on $\mathcal{L}_n \cdot \mathcal{L}_n$.

Proposition 2.1. *The polynomial P_n interpolates Ω_M at the table A_n in the sense that*

$$\Omega_M(z) - P_n(z) = z \omega_n(z) F_n(z) \quad (2.2)$$

and the polynomial Q_n interpolates Ω_M at the table $\tilde{A}_n$ in the sense that

$$\Omega_M(z) - Q_n(z) = \frac{\omega_n(z)}{z^{n+1}} G_n(z). \quad (2.3)$$

Here F_n and G_n are functions which are defined at least for $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ and $\{\alpha_\infty, \alpha_1, \dots, \alpha_n\}$, respectively.

Proof: It can easily be verified by induction that

$$1 + 2 \sum_{k=1}^n \frac{z \omega_{k-1}(z)}{\omega_k(t)} = \frac{t+z}{t-z} \left[1 - \frac{z \omega_n(z)}{t \omega_n(t)} \right] - \frac{z \omega_n(z)}{t \omega_n(t)} \quad (2.4)$$

and

$$\frac{t+z}{t-z} - \left[1 + 2 \sum_{k=1}^n \frac{z \omega_{k-1}(z)}{\omega_k(t)} \right] = \frac{2z \omega_n(z)}{(t-z) \omega_n(t)}. \quad (2.5)$$

Similarly (e.g. by applying the substar transform) we get

$$1 - 2 \sum_{k=1}^n \alpha_k \frac{t^k \omega_{k-1}(z)}{z^k \omega_k(t)} = -\frac{t+z}{t-z} \left[1 - \frac{t^{n+1} \omega_n(z)}{z^{n+1} \omega_n(t)} \right] - \frac{t^{n+1} \omega_n(z)}{z^{n+1} \omega_n(t)} \quad (2.6)$$

and

$$\frac{t+z}{t-z} + \left[1 - 2 \sum_{k=1}^n \alpha_k \frac{t^k \omega_{k-1}(z)}{z^k \omega_k(t)} \right] = \frac{2t^{n+1} \omega_n(z)}{z^n (t-z) \omega_n(t)}. \quad (2.7)$$

Operating on Equations (2.5) and (2.7) by M we get

$$\Omega_M(z) - [\mu_0 + 2 \sum_{k=1}^n \mu_k z \omega_{k-1}(z)] = 2z \omega_n(z) M \left[\frac{1}{(t-z) \omega_n(t)} \right] \quad (2.8)$$

and

$$\Omega_M(z) - [\nu_0 + 2 \sum_{k=1}^n \nu_k \frac{\omega_{k-1}(z)}{z^k}] = 2 \frac{\omega_n(z)}{z^{n+1}} M \left[\frac{z t^{n+1}}{(t-z) \omega_n(t)} \right]. \quad (2.9)$$

The left sides of (2.8) and (2.9) are the left sides of (2.2) and (2.3). We introduce

$$F_n(z) = 2M\left[\frac{1}{(t-z)\omega_n(t)}\right], G_n(z) = 2M\left[\frac{zt^{n+1}}{(t-z)\omega_n(t)}\right]. \quad (2.10)$$

It follows from the assumption of M being defined on $\tilde{\mathcal{L}} \cdot \tilde{\mathcal{L}}$ (see Section 1) that $F_n(z)$ and $G_n(z)$ are defined (as finite numbers) for $z = \alpha_1, \dots, \alpha_n$. Also both $F_n(0) = 2M\left[\frac{1}{t\omega_n(t)}\right]$ and $G_n(\infty) = 2M\left[\left(\frac{t^{n+1}}{(\zeta(t-1)\omega_n(t))}\right)_{\zeta=0}\right]$ are defined by assumption. $\square$

We shall make use of alternative expressions for $\psi_n(z)$ to the one given in the definition (1.11).

Lemma 2.2. *Let $m = 0, 1, \dots, n-1$. Then we may write*

$$\psi_n(z) = M\left[\frac{t+z}{t-z} \left(\frac{\omega_m(z)}{\omega_m(t)} \varphi_n(t) - \varphi_n(z)\right)\right]. \quad (2.11)$$

Proof: We may write

$$\psi_n(z) - M\left[\frac{t+z}{t-z} \left(\frac{\omega_m(z)}{\omega_m(t)} \varphi_n(t) - \varphi_n(z)\right)\right] = M\left[\frac{(t+z)(\omega_m(t) - \omega_m(z))\varphi_n(t)}{(t-z)\omega_m(t)}\right]. \quad (2.12)$$

The argument of M on the right side is of the form $\frac{R_m(t)\varphi_n(t)}{\omega_m(t)}$, $R_m \in \Pi_m$, hence the functional value is zero by orthogonality. $\square$

We shall use the following notation:

$$\Delta_{n,s} = \psi_n(z) + \varphi_n(z) P_s(z), \quad (2.13)$$

$$\tilde{\Delta}_{n,s} = \psi_n(z) + \varphi_n(z) Q_s(z). \quad (2.14)$$

We recall that p_n and q_n denote the numerator polynomials of φ_n and ψ_n , cf. (1.10) and (1.12).

Proposition 2.3. *For $m < n$ and arbitrary s the following formula is valid:*

$$q_n(z) + p_n(z) P_s(z) = 2zM\left[\frac{\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - \omega_s(z)\omega_n(t)\omega_m(t)p_n(z)}{(t-z)\omega_s(t)\omega_n(t)\omega_m(t)}\right]. \quad (2.15)$$

Proof: Taking into account (1.20), (1.22) and (2.11) we may write for $m < n$:

$$\Delta_{n,s} = M\left[\frac{t+z}{t-z}\left(\frac{\omega_m(z)}{\omega_m(t)}\varphi_n(t) - \varphi_m(z)\right) + \varphi_n(z)\left(1 + 2\sum_{k=1}^s \frac{z\omega_{k-1}(z)}{\omega_k(t)}\right)\right] \quad (2.16)$$

and by using (2.4) with n replaced by s in the summation we get

$$\Delta_{n,s} = M\left[\frac{(t+z)\omega_m(z)\varphi_n(t)}{(t-z)\omega_m(t)} - \frac{2z\omega_s(z)\varphi_n(z)}{(t-z)\omega_s(t)}\right]. \quad (2.17)$$

Substitution of the expansion (1.10) for φ_n then yields

$$\Delta_{n,s} = M\left[\frac{(t+z)\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - 2z\omega_s(z)\omega_n(t)\omega_m(t)p_n(z)}{(t-z)\omega_s(t)\omega_n(t)\omega_m(t)\omega_n(z)}\right]. \quad (2.18)$$

This may again be written as

$$\begin{aligned} \Delta_{n,s} &= M\left[\frac{2z[\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - \omega_s(z)\omega_n(t)\omega_m(t)p_n(z)]}{(t-z)\omega_s(t)\omega_n(t)\omega_m(t)\omega_n(z)}\right] \\ &+ \omega_m(z) M\left[\frac{p_n(t)}{\omega_n(t)\omega_m(t)}\right]. \end{aligned} \quad (2.19)$$

The argument of M in the last term may be written as $\varphi_n(t) \frac{1}{\omega_m(t)}$, hence the last term is zero since $m < n$. (Note that the argument of the first term to the right in (2.19) belongs to the domain of definition of M , since the argument in the last term in (2.19) and the argument in (2.17) (hence also in (2.16) does.) Thus

$$\Delta_{n,s} = \frac{2z}{\omega_n(z)} M\left[\frac{\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - \omega_s(z)\omega_n(t)\omega_m(t)p_n(z)}{(t-z)\omega_s(t)\omega_n(t)\omega_m(t)}\right]. \quad (2.20)$$

Multiplication by $\omega_n(z)$ gives (2.15). □

Proposition 2.4. For $m < n$ and arbitrary s the following formula is valid:

$$\begin{aligned} &q_n(z) + p_n(z)Q_s(z) \\ &= \frac{2}{z^s} M\left[\frac{tz^s\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - t^{s+1}\omega_s(z)\omega_n(t)\omega_m(t)p_n(z)}{(t-z)\omega_s(t)\omega_n(t)\omega_m(t)}\right]. \end{aligned} \quad (2.21)$$

Proof: Using (1.21) and (1.23) we get for $m < n$:

$$\tilde{\Delta}_{n,s} = M\left[\frac{t+z}{t-z}\left(\frac{\omega_m(z)}{\omega_m(t)}\varphi_n(t) - \varphi_n(z)\right) + (-1 + 2\sum_{k=1}^s \alpha_k \frac{t^k\omega_{k-1}(z)}{z^k\omega_k(t)})\right] \quad (2.22)$$

and by (2.6) with n replaced by s in the summation this yields

$$\tilde{\Delta}_{n,s} = M\left[\frac{(t+z)\omega_m(z)\varphi_n(t)}{(t-z)\omega_m(t)} - \frac{2t^{s+1}\omega_s(z)\varphi_n(z)}{z^s(t-z)\omega_s(t)}\right]. \quad (2.23)$$

This may be written as

$$\tilde{\Delta}_{n,s} = M\left[\frac{(t+z)z^s\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - 2t^{s+1}\omega_s(z)\omega_n(t)\omega_m(t)p_n(z)}{z^s(t-z)\omega_s(t)\omega_n(t)\omega_m(t)\omega_n(z)}\right]. \quad (2.24)$$

Rearranging we get

$$\begin{aligned} \tilde{\Delta}_{n,s} = M\left[\frac{2[tz^s\omega_s(t)\omega_n(z)\omega_m(z)p_n(t) - t^{s+1}\omega_s(t)\omega_n(t)\omega_m(t)p_n(z)]}{z^s(t-z)\omega_s(t)\omega_n(t)\omega_m(t)\omega_n(z)}\right] \\ - \omega_m(z) M\left[\frac{p_n(t)}{\omega_n(t)\omega_m(t)}\right]. \end{aligned} \quad (2.25)$$

The last term is zero (cf. the proof of Proposition 2.3), and so (2.4) follows by multiplication by $\omega_n(z)$. 21

□

For each n we define interpolation tables $B_{2n-1}^{(n)}$ and $\tilde{B}_{2n-1}^{(n)}$ by

$$B_{2n-1}^{(n)} = \{\alpha_0, \beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_{2n-1}^{(n)}\} \quad (2.26)$$

$$\tilde{B}_{2n-1}^{(n)} = \{\alpha_\infty, \beta_1^{(n)}, \beta_2^{(n)}, \dots, \beta_{2n-1}^{(n)}\}, \quad (2.27)$$

where

$$\beta_k^{(n)} = \alpha_k \text{ for } 0 \leq k \leq n, \quad \beta_k^{(n)} = \alpha_{k-n} \text{ for } n < k \leq 2n-1. \quad (2.28)$$

We further define

$$\omega_0^{(n)} = 1, \quad \omega_k^{(n)}(z) = \prod_{j=1}^k (z - \beta_j^{(n)}) \text{ for } k = 1, 2, \dots, 2n-1. \quad (2.29)$$

Note that

$$\omega_k^{(n)}(z) = \omega_n(z)\omega_{k-n}(z) \text{ for } n < k \leq 2n-1. \quad (2.30)$$

We also define moments $\mu_k^{(n)}$, $\nu_k^{(n)}$ by

$$\mu_k^{(n)} = M\left[\frac{1}{\omega_k^{(n)}(t)}\right], \quad k = 0, 1, 2, \dots, 2n-1, \quad (2.31)$$

$$\nu_0^{(n)} = -\mu_0^{(n)}, \quad \nu_k^{(n)} = M\left[\frac{\alpha_k t^k}{\omega_k^{(n)}(t)}\right], \quad k = 1, 2, \dots, 2n-1. \quad (2.32)$$

Then $\mu_k^{(n)} = \mu_k$, $\nu_k^{(n)} = \nu_k$ for $k = 0, 1, \dots, n$. Finally we introduce polynomials $P_{2n-1}^{(n)}$ and $Q_{2n-1}^{(n)}$ defined by

$$P_{2n-1}^{(n)}(z) = \mu_0^{(n)} + 2 \sum_{k=1}^{2n-1} \mu_k^{(n)} z \omega_{k-1}^{(n)}(z), \quad (2.33)$$

$$Q_{2n-1}^{(n)}(z) = \nu_0^{(n)} + 2 \sum_{k=1}^{2n-1} \nu_k^{(n)} \frac{\omega_{k-1}^{(n)}(z)}{z^k}. \quad (2.34)$$

Proposition 2.5. *The polynomial $P_{2n-1}^{(n)}(z)$ interpolates $\Omega_M(z)$ in the sense that*

$$\Omega_M(z) - P_{2n-1}^{(n)}(z) = z \omega_n(z) \omega_{n-1}(z) f_{2n-1}(z) \quad (2.35)$$

and the polynomial $Q_{2n-1}^{(n)}(z)$ interpolates $\Omega_M(z)$ in the sense that

$$\Omega_M(z) - Q_{2n-1}^{(n)}(z) = \frac{\omega_n(z) \omega_{n-1}(z)}{z^{2n}} g_{2n-1}(z). \quad (2.36)$$

Here $f_{2n-1}(z)$ has finite values for $z = \alpha_0, \alpha_1, \dots, \alpha_n$, and $g_{2n-1}(z)$ has finite values for $z = \alpha_\infty, \alpha_1, \dots, \alpha_n$.

Proof: This follows by applying Proposition 2.1 to the tables $B_{2n-1}^{(n)}$ and $\tilde{B}_{2n-1}^{(n)}$, taking (2.30) into account. □

Proposition 2.6. *The following formula is valid:*

$$q_n(z) + p_n(z) P_{2n-1}^{(n)}(z) = z \omega_n(z) \omega_{n-1}(z) F_{2n-1}(z), \quad (2.37)$$

where $F_{2n-1}(z)$ has finite values for $z = \alpha_0, \alpha_1, \dots, \alpha_n$.

Proof: It follows by applying Proposition 2.3 to the table $B_{2n-1}^{(n)}$, setting $m = n - 1$, $s = 2n - 1$ and taking into account (2.30), that

$$q_n(z) + p_n(z) P_{2n-1}^{(n)}(z) = 2z\omega_n(z)\omega_{n-1}(z)M\left[\frac{p_n(t) - p_n(z)}{(t-z)\omega_n(t)\omega_{n-1}(t)}\right]. \quad (2.38)$$

The function $F_{2n-1}(z) = 2M\left[\frac{p_n(t) - p_n(z)}{(t-z)\omega_n(t)\omega_{n-1}(t)}\right]$ is defined and finite at $z = \alpha_0, \alpha_1, \dots, \alpha_n$ by assumption, the expression in brackets belonging to the domain of definition of M . what if $z = \alpha_k$, $ch(2.10)$

□

Proposition 2.7. *The following formula is valid:*

$$q_n(z) + p_n(z)Q_{2n-1}^{(n)}(z) = \frac{\omega_n(z)}{z} G_{2n-1}(z), \quad (2.39)$$

where $G_{2n-1}(z)$ has finite values for $z = \alpha_\infty, \alpha_1, \dots, \alpha_n$.

Proof: It follows by applying Proposition 2.4 to the table $\tilde{B}_{2n-1}^{(n)}$, setting $m = 0$, $s = 2n - 1$ and taking into account (2.30), that

$$q_n(z) + p_n(z)Q_{2n-1}^{(n)} = 2\frac{\omega_n(z)}{z} M\left[\frac{tz^{2n-1}\omega_{n-1}(t)p_n(t) - t^{2n}\omega_{n-1}(z)p_n(z)}{(t-z)\omega_n(t)\omega_{n-1}(t)z^{2n-2}}\right]. \quad (2.40)$$

The function $G_{2n-1}(z) = 2M\left[\frac{tz^{2n-1}\omega_{n-1}(t)p_n(t) - t^{2n}\omega_{n-1}(z)p_n(z)}{(t-z)\omega_n(t)\omega_{n-1}(t)z^{2n-2}}\right]$ is defined and finite for $z = \alpha_1, \dots, \alpha_n$. Furthermore

$H_{2n-1}(\zeta) = G_{2n-1}(\zeta^{-1}) = 2M\left[\frac{t\omega_{n-1}(t)p_n(t) - \zeta^{2n-1}t^{2n}\omega_{n-1}(\zeta^{-1})p_n(\zeta^{-1})}{(t\zeta - 1)\omega_n(t)\omega_{n-1}(t)}\right]$ is finite for $\zeta = 0$, since argument for M in this expression belong to the domain of definition of M for $\zeta = 0$. the

□

Theorem 2.8. *The rational function $\frac{q_n(z)}{p_n(z)} = \frac{\psi_n(z)}{\varphi_n(z)}$ interpolates $-\Omega_M(z)$ linearly at the table $\{\alpha_0, \alpha_\infty, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \dots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n\}$ in the sense that*

$$q_n(z) + p_n(z)\Omega_M(z) = A_n(z)z\omega_n(z)\omega_{n-1}(z) \quad (2.41)$$

and

$$q_n(z) + p_n(z)\Omega_M(z) = B_n(z) \cdot \frac{\omega_n(z)}{z} \quad (2.42)$$

where $A_n(z)$ and $B_n(z)$ are functions such that $A_n(z)$ is finite for $z = \alpha_0, \alpha_1, \dots, \alpha_n$ and $B_n(z)$ is finite for $z = \alpha_\infty, \alpha_1, \dots, \alpha_n$.

Proof: We may write

$$q_n(z) + p_n(z)\Omega_M(z) = q_n(z) + p_n(z)P_{2n-1}^{(n)}(z) + p_n(z)[\Omega_M(z) - P_{2n-1}^{(n)}(z)]. \quad (2.43)$$

Formula (2.41) now follows from (2.35) and (2.37). Similarly we may write

$$q_n(z) + p_n(z)\Omega_M(z) = q_n(z) + p_n(z)Q_{2n-1}^{(n)}(z) + p_n(z)[\Omega_M(z) - Q_{2n-1}^{(n)}(z)] \quad (2.44)$$

and so Formula (2.42) follows from (2.36) and (2.39). □

Remark. The rational function $\frac{p_n(z)}{q_n(z)}$ satisfies $2n+1$ interpolation conditions for $-\Omega_M(z)$, while the degrees of the polynomials $p_n(z)$ and $q_n(z)$ are both at most n . Thus $\frac{p_n(z)}{q_n(z)}$ is a multipoint Padé approximant to $-\Omega_M(z)$.

Instead of introducing the new tables $B_{2n-1}^{(n)}$ and $\tilde{B}_{2n-1}^{(n)}$ for each n we might introduce one new table $B = \{\alpha_0, \alpha_\infty, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \dots, \alpha_n, \alpha_n, \dots\}$. Orthogonality would be defined in terms of the table $A = \{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n, \dots\}$, interpolation would be at points of B . Moments, however, would have to be defined in terms of the tables $B_{2n-1}^{(n)}, \tilde{B}_{2n-1}^{(n)}$.

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