On the Halton sequence and its scramblings

Bart Vandewoestyne*, Ronald Cools

Department of Computer Science
KULeuven

May 18, 2005
The Halton sequence
The Halton sequence

Scrambling the Halton sequence
The Halton sequence

Scrambling the Halton sequence

Scrambled Halton: LDS?
The Halton sequence

Scrambling the Halton sequence

Scrambled Halton: LDS?

A new and simple scrambling
On the Halton sequence and its scramblings

Outline

The Halton sequence

Scrambling the Halton sequence

Scrambled Halton: LDS?

A new and simple scrambling

Conclusions
The Halton sequence

Construction
Problem

Scrambling the Halton sequence

Scrambled Halton: LDS?

A new and simple scrambling

Conclusions
Halton: construction

Let $b \geq 2$ be an integer, then any integer $n \geq 0$ can be written in the form

$$n = d_j b^j + \cdots + d_2 b^2 + d_1 b + d_0, \quad 0 \leq d_i < b.$$
Halton: construction

- Let $b \geq 2$ be an integer, then any integer $n \geq 0$ can be written in the form

$$n = d_j b^j + \cdots + d_2 b^2 + d_1 b + d_0, \quad 0 \leq d_i < b.$$

- The radical inverse function $\phi_b(n)$ for base $b$ is then defined by

$$\phi_b(n) = \frac{d_0}{b} + \frac{d_1}{b^2} + \cdots + \frac{d_j}{b^{j+1}}.$$
Halton: construction

- Let $b \geq 2$ be an integer, then any integer $n \geq 0$ can be written in the form

$$n = d_j b^j + \cdots + d_2 b^2 + d_1 b + d_0, \quad 0 \leq d_i < b.$$  

- The radical inverse function $\phi_b(n)$ for base $b$ is then defined by

$$\phi_b(n) = \frac{d_0}{b} + \frac{d_1}{b^2} + \cdots + \frac{d_j}{b^{j+1}}.$$  

- It’s like ‘mirroring around the decimal point’:

$$d_j \ldots d_2 d_1 d_0 \Rightarrow 0.d_0 d_1 \ldots d_j$$
Halton: construction

Using these definitions, the *van der Corput* sequence in base $b$ is the one-dimensional point set

$$\{\phi_b(n)\}_{n=0}^{\infty} = \{\phi_b(0), \phi_b(1), \phi_b(2), \ldots\}.$$
Halton: construction

- Using these definitions, the *van der Corput* sequence in base $b$ is the one-dimensional point set

$$\{\phi_b(n)\}_{n=0}^{\infty} = \{\phi_b(0), \phi_b(1), \phi_b(2), \ldots\}.$$  

- Halton extended this definition to the $s$-dimensional sequence $\{x_i\}$, defining

$$x_n = (\phi_{b_1}(n), \ldots, \phi_{b_s}(n)), \quad n = 0, 1, \ldots$$

with $b_1, \ldots, b_s > 1$ and pairwise prime.
Halton: discrepancy bounds

**Theorem (Halton 1960)**

The Halton sequence is a low-discrepancy sequence with

\[
D^*_N(P) < \prod_{i=1}^{s} \frac{3b_i - 2 (\ln N)^s}{\ln b_i N},
\]

**Theorem (Niederreiter 1992)**

The Halton sequence is a low-discrepancy sequence with

\[
D^*_N(P) < \frac{s}{N} + \frac{1}{N} \prod_{i=1}^{s} \left( \frac{b_i - 1}{2 \ln b_i} \ln N + \frac{b_i + 1}{2} \right)
\]
Halton: problem

Consider the following projections of the first 200 Halton points:

Figure: Good :-)  
Figure: Bad! :-(

A possible solution for this is to \textit{scramble} the Halton sequence.
Scrambled Halton: example

Figure: before scrambling

Figure: after scrambling
Scrambling the Halton sequence

Construction

Examples

Scrambled Halton: LDS?

A new and simple scrambling

Conclusions
Scrambled Halton: construction

- Instead of using the radical inverse, use the scrambled radical inverse

\[ S_b(n) = \frac{\pi_b(d_0)}{b} + \frac{\pi_b(d_1)}{b^2} + \cdots + \frac{\pi_b(d_j)}{b^{j+1}}, \]

where \( \pi_b \) is a permutation on the digits \((0,1,\ldots,b-1)\) which holds 0 fixed.

- The scrambled Halton sequence is then given by

\[ x_n = (S_{b_1}(n), \ldots, S_{b_s}(n)), \quad n = 0, 1, \ldots \]

- Note that there are \( b! \) possibilities to choose from for each \( \pi_b \)

\( \Rightarrow \) many many scramblings... but which one is the best?
Permutations by Warnock (1972)

Warnock used the \textit{folded radical inverse} function

\[
\psi_b(n) = \frac{(d_0 + 0) \mod b}{b} + \frac{(d_1 + 1) \mod b}{b^2} + \cdots + \frac{(d_j + j) \mod b}{b^{j+1}} + \cdots
\]

to define a scrambled version of the Halton sequence.
Permutations by Braaten and Weller (1979)

- Algorithm: having picked $\pi_b(1) \ldots \pi_b(j)$, choose $\pi_b(j + 1)$ as to minimize the discrepancy of the set

\[
\left\{ \frac{\pi_b(1)}{b}, \ldots, \frac{\pi_b(j)}{b}, \frac{\pi_b(j + 1)}{b} \right\}. 
\]

- This is a lot of work!

- Only first 16 permutations are published...
Mascagni and Chi considered the linear scrambling

$$\pi_{b_i}(d_j) = w_i d_j \mod b_i,$$

and searched for optimal $w_i$.

Empirical verification of ‘goodness’ by calculating one test-integral.
Permutations by ...

Other suggestions found in the literature:

- Faure (1992)
- Warnock PhiCf (1995)
- Tuffin (1996)

General remarks:

- Most of these require quite some work to set up the $\pi_b$.
- Some have only published the permutations up to 16 dimensions.
The Halton sequence

Scrambling the Halton sequence

**Scrambled Halton: LDS?**

- Another point of view
- Halton’s proof
- Niederreiter’s proof

A new and simple scrambling

Conclusions
The following question arises:

"Is a scrambled Halton sequence still a low-discrepancy sequence?"

Widely accepted, but nowhere formally proved.
Braaten and Weller (1979)

- Citing their paper:
  
  “We have proven that Halton’s upper bound continues to hold for the family of sequences described below.”
Braaten and Weller (1979)

- Citing their paper:
  
  "We have proven that Halton’s upper bound continues to hold for the family of sequences described below."

- Unpublished.
"It is generally believed that any permutation of the coefficients $a_i(j, n)$ in (...) gives the discrepancy bound (...). The constant implied from the discrepancy bound for the generalized Halton sequence will certainly depend on the particular permutations selected; however, theoretical formulae showing this are not available. We are not aware of a proof that for $s > 2$ sequence (...) with $\phi_{b_j(n)}$ defined as (...) has discrepancy bound $O((\log N)^s / N)$. Proof that the generalized Halton sequence has the discrepancy bound (...) for $s = 2$ was given by Faure [1986]."
Let’s take a look at scrambling from another point of view...
Traditional approach

1. Start from the digit representation of an integer $n_{\text{original}}$ in base $b$:

$$n_{\text{original}} = d_M \ldots d_2 d_1 d_0$$
Traditional approach

1. Start from the digit representation of an integer \( n_{\text{original}} \) in base \( b \):

\[
n_{\text{original}} = d_M \ldots d_2 d_1 d_0
\]

2. Apply the radical inverse function:

\[
\phi_b(n_{\text{original}}) = 0.d_0d_1d_2 \ldots d_M
\]
Traditional approach

1. Start from the digit representation of an integer $n_{original}$ in base $b$:

   $n_{original} = d_M \ldots d_2 d_1 d_0$

2. Apply the radical inverse function:

   $\phi_b(n_{original}) = 0.d_0 d_1 d_2 \ldots d_M$

3. Finally scramble the digits to get

   $S_b(n_{original}) = 0.\pi(d_0)\pi(d_1)\pi(d_2)\ldots\pi(d_M)$
Alternative approach

1. Start from the digit representation of an integer $n_{\text{original}}$ in base $b$:

$$n_{\text{original}} = d_M \ldots d_2 d_1 d_0$$
Alternative approach

1. Start from the digit representation of an integer $n_{original}$ in base $b$:

   $$n_{original} = d_M \ldots d_2 d_1 d_0$$

2. Apply the permutation to the digits, to get another integer $n_{permuted}$

   $$\pi(n_{original}) = n_{permuted} = \pi(d_M) \ldots \pi(d_2) \pi(d_1) \pi(d_0)$$
Alternative approach

1. Start from the digit representation of an integer $n_{\text{original}}$ in base $b$:
   \[ n_{\text{original}} = d_M \ldots d_2 d_1 d_0 \]

2. Apply the permutation to the digits, to get another integer $n_{\text{permuted}}$:
   \[ \pi(n_{\text{original}}) = n_{\text{permuted}} = \pi(d_M) \ldots \pi(d_2) \pi(d_1) \pi(d_0) \]

3. Finally apply the standard radical inverse to $n_{\text{permuted}}$, resulting in
   \[ S_b(n_{\text{original}}) = \phi(n_{\text{permuted}}) = 0.\pi(d_0) \pi(d_1) \ldots \pi(d_M) \]
Example: traditional approach

\[ n_{\text{original}} = 123456 \]
\[ \downarrow \quad \text{radical inverse} \]
\[ \phi_{10}(n_{\text{original}}) = 0.654321 \]
\[ \downarrow \quad \text{permute} \]
\[ \phi_{10}(n_{\text{permuted}}) = S_{10}(n_{\text{original}}) = 0.396718 \]

Note: permutation used is

\[ \pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 8 & 1 & 7 & 6 & 9 & 3 & 4 & 2 & 5 \end{pmatrix} \].
Example: alternative approach

\[
\begin{align*}
  n_{\text{original}} &= 123456 \\
  \downarrow & \text{ permute} \\
  n_{\text{permuted}} &= \pi(n_{\text{original}}) = 817693 \\
  \downarrow & \text{ radical inverse} \\
  \phi_{10}(n_{\text{permuted}}) = S_{10}(n_{\text{original}}) &= 0.396718
\end{align*}
\]

Note: permutation used is

\[
\pi = \begin{pmatrix}
  0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  0 & 8 & 1 & 7 & 6 & 9 & 3 & 4 & 2 & 5
\end{pmatrix}.
\]
What does this tell us?

- If we generate a certain dimension of our scrambled Halton sequence with the integers

\[ \{1, \ldots, n\}, \]
What does this tell us?

- If we generate a certain dimension of our scrambled Halton sequence with the integers

{1, \ldots, n},

- these integers get permuted to

{\pi(1), \ldots, \pi(n)},

max_{\{\pi(1), \ldots, \pi(n)\}} < b^{M+1},
What does this tell us?

- If we generate a certain dimension of our scrambled Halton sequence with the integers

\[ \{1, \ldots, n\}, \]

- these integers get permuted to

\[ \{\pi(1), \ldots, \pi(n)\}, \]

- with

\[ \max\{\pi(1), \ldots, \pi(n)\} < b^{M+1}, \]
What does this tell us?

- If we generate a certain dimension of our scrambled Halton sequence with the integers
  \[ \{1, \ldots, n\}, \]

- these integers get permuted to
  \[ \{\pi(1), \ldots, \pi(n)\}, \]

- with
  \[ \max\{\pi(1), \ldots, \pi(n)\} < b^{M+1}, \]

- and we use these new integers to do a standard radical inverse. So . . .
...the $j$-th coordinates of the points of a scrambled Halton sequence are a **subset** of a longer, unscrambled set of coordinates generated by integers from $\{1, \ldots, \max\{\pi(1), \ldots, \pi(n)\}\}$

![Diagram showing the relationship between original and permuted indices.](image)
Example:

The first 10 scrambled coordinates in base 3 are a subset of the first 20 unscrambled coordinates

\[
\pi = (0 \ 2 \ 1) \text{ so that } \\
\{\pi(1), \ldots, \pi(10)\} = \{2, 1, 6, 8, 7, 3, 5, 4, 18, 20\}
\]
Example: Braaten and Weller’s permutations (base 7)
Example: Braaten and Weller’s permutations (base 7)
Is a scrambled Halton sequence still a low-discrepancy sequence?

Yes, the scrambled Halton sequence is a low-discrepancy sequence for which . . .

- . . . Halton’s discrepancy bound remains valid:
  \[
  D^*_N(P) < \prod_{i=1}^{s} \frac{3b_i - 2 (\ln N)^s}{\ln b_i} \frac{\ln N}{N}.
  \]

- . . . Niederreiter’s discrepancy bound remains valid:
  \[
  D^*_N(P) < \frac{s}{N} + \frac{1}{N} \prod_{i=1}^{s} \left( \frac{b_i - 1}{2 \ln b_i} \ln N + \frac{b_i + 1}{2} \right)
  \]
Key-idea of Halton’s proof

Which \( n_{\text{original}} \) will lead to a radical inverse value inside the rectangle \( B \)?

\[
R(B; P) = \frac{A(B; P)}{N} - \text{Volume}(B)
\]

\( \Rightarrow \) Count the points in the integer domain instead of in \([0, 1)\).
Counting in the ‘integer domain’: non-permuted version

From Halton’s paper and by intuition we know:

The number of integers in \{1, \ldots, N\} which are congruent modulo \(b^m\) (e.g. 10, 10^2, 10^3, \ldots) to a certain number \(p\) (consisting of \(m\) digits) from that set is

\[ \left\lfloor \frac{N}{b^m} \right\rfloor + h \]

with \(h\) equal to 0 or 1.
The numbers from \( \{1, 2, \ldots, 44\} \) which are congruent to 3 modulo 10 are

\[
3, 13, 23, 33, 43
\]

\( \rightarrow \) there are \( \lfloor 44/10^1 \rfloor + 1 = 5 \) of them.
Counting in the ‘integer domain’: permuted version

Because permuting one integer to another is a bijective operation, we also know:

The number of integers in \( \{\pi(1), \ldots, \pi(N)\} \) which are congruent modulo \( b^m \) to a certain number \( \pi(p) \) (consisting of \( m \) digits) from that set is also

\[ \left\lfloor N/b^m \right\rfloor + h \]

with \( h \) equal to 0 or 1.
Permuted integer count: example

- The numbers from \( \{\pi(1), \pi(2), \ldots, \pi(44)\} \) which are congruent to \( \pi(3) = 4 \) modulo 10 are

\[
\pi(3) = 4, \quad \pi(13) = 64, \quad \pi(23) = 54, \quad \pi(33) = 44, \quad \pi(43) = 34
\]

- there are also 5 of them.
Conclusion:

- The integer count remains the same
- The rest of the proof also remains the same
- The star-discrepancy bound remains
On the Halton sequence and its scramblings

Scrambled Halton: LDS?

Niederreiter’s proof

Key-idea of Niederreiter’s proof

Proof by induction, with initial-step property:

Figure: Halton

Figure: Scrambled Halton
Conclusion:

- The induction basis is still valid
- The rest of the proof also remains the same
- The star-discrepancy bound remains
On the Halton sequence and its scramblings

A new and simple scrambling

The Halton sequence

Scrambling the Halton sequence

Scrambled Halton: LDS?

A new and simple scrambling

The reverse Halton sequence

Compared to deterministic scrambling

Compared to randomized Halton sequences

Conclusions
A new proposal: permutation reverse

Why not just try these permutations and see how well they perform?

| $\pi_2$ | (0 1)  |
| $\pi_3$ | (0 2 1) |
| $\pi_5$ | (0 4 3 2 1) |
| ... | ... |
| $\pi_b$ | (0 $b-1$ $b-2$ ... 1) |
| ... | ... |

Table: ‘Reverse’ permutations

Easy to generate!
'Reverse’ permutations for the Halton sequence

Note that a Halton sequence with reverse permutations still suffers from a certain degree of correlation. Take

\[ \delta(b - a_i) = \begin{cases} 
0 & \text{if } a_i = 0, \\
 b - a_i & \text{if } a_i \neq 0
\end{cases}, \quad \Theta \left( \frac{1}{b^i} \right) = \begin{cases} 
0 & \text{if } a_i = 0, \\
 \frac{1}{b^i} & \text{if } a_i \neq 0
\end{cases}, \]

then the scrambled radical inverse function for the reverse Halton sequence can be written as

\[ S_b(n) = \frac{\delta(b - a_0)}{b^1} + \cdots + \frac{\delta(b - a_M)}{b^{M+1}} \]

\[ = \Theta \left( \frac{1}{b^0} \right) + \cdots + \Theta \left( \frac{1}{b^M} \right) - \left( \frac{a_0}{b^1} + \cdots + \frac{a_M}{b^{M+1}} \right), \]
‘Reverse’ permutations for the Halton sequence

\[ S_b(n) = \Theta \left( \frac{1}{b^0} \right) + \cdots + \Theta \left( \frac{1}{b^M} \right) - \left( \frac{a_0}{b^1} + \cdots + \frac{a_M}{b^{M+1}} \right). \]

- If \( n \) contains no zeros, this is simply a shift.
- If \( n \) contains zeros in it’s digit representation, the shift will vary.
- How much we shift depends on the location and the number of zero digits.
- Groups of \( n \)-values having the same number of zeros at the same places will shift by the same amount.
- Actually, it is a linear scrambling with\[ \pi_{b_i}(d_j) = (b_i - 1)d_j \mod b_i. \]
‘Reverse’ permutations for the Halton sequence

Although there is still a certain degree of correlation, $L_2$ discrepancy of the reverse Halton sequence is low compared to other deterministic Halton scramblings...
The Halton sequence

Scrambling the Halton sequence

Scrambled Halton: LDS?

A new and simple scrambling
   - The reverse Halton sequence
   - Compared to deterministic scrambling
   - Compared to randomized Halton sequences

Conclusions
On the Halton sequence and its scramblings
A new and simple scrambling
Compared to deterministic scrambling

$T^*_N$ for 16-dimensional sequences

- Halton
- BW
- PhiCf
- MCL∗
- Reverse
- RMS discrepancy for random sequence

Number of points
Star discrepancy
Halton
FAU
MCL
WA1
CHI
Reverse
RMS discrepancy for random sequence
On the Halton sequence and its scramblings

A new and simple scrambling

Compared to deterministic scrambling

$T^*_N$ for 32-dimensional sequences

<table>
<thead>
<tr>
<th>Number of points</th>
<th>Star discrepancy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Halton</td>
<td>WA1</td>
</tr>
<tr>
<td>FAU</td>
<td>CHI</td>
</tr>
<tr>
<td>Reverse</td>
<td>RMS discrepancy for random sequence</td>
</tr>
</tbody>
</table>

Graph showing the comparison of different sequences in terms of star discrepancy and number of points.
On the Halton sequence and its scramblings
- A new and simple scrambling
- Compared to randomized Halton sequences

The Halton sequence

Scrambling the Halton sequence

Scrambled Halton: LDS?

A new and simple scrambling
- The reverse Halton sequence
- Compared to deterministic scrambling
- Compared to randomized Halton sequences

Conclusions
On the Halton sequence and its scramblings
“A new and simple scrambling
Compared to randomized Halton sequences

Randomization: philosophy

- Randomization ≠ deterministic scrambling
- Randomized instances of the Halton sequence cannot be expressed by means of

\[ S_b(n) = \frac{\pi_b(d_0)}{b} + \frac{\pi_b(d_1)}{b^2} + \cdots + \frac{\pi_b(d_j)}{b^{j+1}} \]

- What randomizations did we look at?
Cranley Patterson rotations

\[ P_{\text{shifted}} = \{(x + \Delta) \mod 1 : x \in P_{\text{original}}\}, \]

with \( \Delta \) a uniform random number over \([0, 1)^s\).
Random-start Halton sequence

Instead of starting with $n^{\text{start}} = n = 0$ to generate each coordinate in

$$x_n = (\phi_{b_1}(n), \ldots, \phi_{b_s}(n)), \quad n = 0, 1, \ldots$$

start from a random $n_i^{\text{start}}$ for each coordinate

$$x_n = (\phi_{b_1}(n_1^{\text{start}} + n), \ldots, \phi_{b_s}(n_s^{\text{start}} + n)). \quad n = 0, 1, \ldots$$
Shuffled-Halton ‘sequences’

Randomly permute the coordinates of each dimension

number of points

dimension
On the Halton sequence and its scramblings

A new and simple scrambling

Compared to randomized Halton sequences

$T^*_N$ of the reverse Halton sequence vs. randomized versions (on average)

![Figure: 16D](image1.png)

![Figure: 32D](image2.png)
On the Halton sequence and its scramblings

- A new and simple scrambling
- Compared to randomized Halton sequences

$T_N^*$ of the reverse Halton sequence vs. randomized versions (on average)

Figure: 64D
Conclusions

- An alternative point of view on deterministic Halton scrambling.
- A star discrepancy bound for scrambled Halton sequences.
- A comparison in terms of $L_2$-discrepancy of different deterministic Halton scramblings.
- $L_2$-discrepancy of ‘reverse’ is low compared to other deterministic and randomized Halton scramblings.
Thanks for your attention!