Quadrature and orthogonal rational functions

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Abstract

Classical interpolatory or Gaussian quadrature formulas are exact on sets of polynomials. The Szegő quadrature formulas are the analogs for quadrature on the complex unit circle. Here the formulas are exact on sets of Laurent polynomials. In this paper we consider generalizations of these ideas, where the (Laurent) polynomials are replaced by rational functions that have prescribed poles. These quadrature formulas are closely related to certain multipoint rational approximants of Cauchy or Riesz-Herglotz transforms of a (positive or general complex) measure. We consider the construction and properties of these approximants and the corresponding quadrature formulas as well as the convergence and rate of convergence.

1 Introduction

It is an important problem in numerical analysis to compute integrals of the form $\int_a^b f(x)\mathrm{d}\mu(x)$ where μ is in general a complex measure on the interval $[a,b]$ with $-\infty \le a < b \le +\infty$. Most quadrature formulas approximate this integral by a weighted combination of function values: $\sum_{i=1}^n A_{ni} f(\xi_{ni})$. The quadrature parameters are the abscissas or knots $\{\xi_{ni}\}_{i=1}^n$ and the coefficients or weights $\{A_{ni}\}_{i=1}^n$. One objective in constructing quadrature formulas could be to find the quadrature parameters such that the formulas are exact for all functions in a class that is as large as possible.

The most familiar quadrature formulas based on this principle are the Gauss-Christoffel formulas. These formulas choose for a positive measure μ as abscissas the zeros of φ_n, which is the polynomial of degree n orthogonal with respect to the inner product $\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\mathrm{d}\mu(x)$. These zeros are simple and all in the interval (a,b). The weights are the so-called Christoffel numbers. They are positive and are constructed in such a way that the quadrature formula is exact for all $f \in \Pi_{2n-1}$, i.e., for any polynomial of degree at most $2n-1$. These Gauss-Christoffel formulas are optimal in the sense that it is impossible to construct

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an n-point formula that is exact in Π_k with $k \geq 2n$. For a survey, see for example [24]. The study of such quadrature formulas was partly motivated by the role they played in the solution of the corresponding Stieltjes-Markov moment problem. That is, given a sequence of complex numbers c_k, find a positive or complex measure μ such that $c_k = \int_a^b x^k \mathrm{d}\mu(x)$, $k = 0, 1, \ldots$. For complex measures see [40, 30].

In this survey, it is our intention to concentrate on the computation of integrals of the form $I_\mu\{f\} := \int_{-\pi}^{\pi} f(e^{i\theta})\mathrm{d}\mu(\theta)$ where f is a complex function defined on the unit circle and μ is in general a complex measure on $[-\pi, \pi]$.

The motivation for this problem is that, just like the previous case is related to a Stieltjes moment problem for an interval, this integral can be related to the solution of a trigonometric moment problem when μ is a positive measure. The Stieltjes-Markov moment problems suggested the construction of quadrature formulas in the largest possible subset of polynomials. However, in the case of the trigonometric moment problem, it is very natural to consider Laurent polynomials (L-polynomials) instead. This is motivated by the fact that a function continuous on the unit circle can be uniformly approximated by L-polynomials. Since L-polynomials are rational functions with poles at the origin and at infinity, the step towards a more general situation where the poles are at several other (fixed) positions in the complex plane seems natural. This will give rise to a discussion of orthogonal L-polynomials and orthogonal rational functions (with arbitrary but fixed poles).

The outline of the paper is as follows. First we introduce the basic ideas and techniques by considering the case of Szegő quadrature formulas that are exact in the largest possible sets of Laurent polynomials for integrals with a positive measure on the unit circle. We introduce the rational (two-point Padé or Padé type) approximants, based on orthogonal polynomials, and the error estimates for the rational approximants and for the quadrature. Section 3 introduces the rational variants of these formulas and approximants. Their convergence is established in Section 5. The necessary properties of orthogonal rational functions needed are discussed briefly in Section 4. Next, we discuss the corresponding problems for a complex measure on the unit circle in Section 6. In Section 7 we also discuss the case where the poles of the rational functions are chosen inside the support of the measure. For ideas related to integrals on an interval (compact or not) of the real line we refer to Section 8. Finally, in Section 9 we state some open problems for further research.

Some notation before we start: $\mathbb{C}$ is the complex plane and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We denote the unit circle by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, the open unit circle by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and the external of the closed disk by $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$. For any function f, the para-hermitian conjugate is defined as $f_*(z) = \overline{f(1/\overline{z})}$. The set of polynomials of degree at most $n \geq 0$ is denoted by Π_n, and Π is the set of all polynomials. By $\Lambda_{p,q} = \{\sum_{k=p}^{q} a_k z^k : a_k \in \mathbb{C}\}$ we denote subsets of L-polynomials, and Λ is the set of all L-polynomials. Note that $\Lambda_{0,n} = \Pi_n$. If $P \in \Pi_n \setminus \Pi_{n-1}$ (where $\Pi_{-1} = \varnothing$), then $P^*(z) = z^n P_*(z)$.

2 The unit circle and L-polynomials

A systematic study of quadrature problems for integrals of the form

$$I_\mu\{f\} := \int_{-\pi}^{\pi} f(e^{i\theta})\mathrm{d}\mu(\theta) \tag{2.1}$$

with μ a positive measure was initiated by Jones et al. [35]. We shall introduce the main ingredients here as an introduction to our more general discussion in the following sections.

The quadrature formula has the form (the measure μ_n is discrete with mass A_{nk} at the points ξ_{nk}, $k = 1, \ldots, n$)

$$I_{\mu_n}\{f\} = \sum_{k=1}^{n} A_{nk} f(\xi_{nk}), \tag{2.2}$$

where the abscissas are all simple and on $\mathbb{T}$. The objective is an analogue of the Gauss-Christoffel formula. That is, find knots and weights such that these formulas are exact in the largest possible set of L-polynomials. So we consider here the polynomial space $\mathcal{L}_n = \Pi_n$ and the space of Laurent polynomials $\mathcal{R}_{p,q} = \Lambda_{-p,q}$, where p and q are always assumed to be nonnegative integers. Note that the dimension of $\mathcal{R}_{p,q}$ is $p + q + 1$. It can be shown that for n different points $\xi_{ni} \in \mathbb{T}$, there is no quadrature formula of the form (2.2) that is exact in some $\mathcal{R}_{p,q}$ of dimension $p + q + 1 > 2n - 1$. But there is a quadrature formula that is exact in $\mathcal{R}_{n-1,n-1}$, and this has the maximal possible dimension. All n-point quadrature formulas with this maximal domain of validity can be described with one free parameter $\tau_n \in \mathbb{T}$. The formulas are called Szegő formulas. They can be described as follows. First we need the orthonormal polynomials φ_k, obtained by orthogonalizing $1, z, z^2, \ldots$ with respect to the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mathrm{d}\mu(\theta).$$

The para-orthogonal polynomials are then defined by $Q_n(z;\tau_n) = \varphi_n(z) + \tau_n\varphi_n^*(z)$. Para-orthogonal means that $Q_n \perp \mathcal{L}_{n-1} \cap z\mathcal{L}_{n-1} = \operatorname{span}\{z, \ldots, z^{n-1}\}$ while $\langle Q_n, 1\rangle \neq 0 \neq \langle Q_n, z^n\rangle$. If $\tau_n \in \mathbb{T}$, then it can be shown that $Q_n(z;\tau_n)$ has n simple zeros $\{\xi_{nk}\}_{k=1}^n \subset \mathbb{T}$. These depend on the parameter τ_n. We can use this parameter to place one zero, e.g., ξ_{n1}, in some arbitrary $w \in \mathbb{T}$. The other knots $\{\xi_{nk}\}_{k=2}^n$ are then the zeros of $k_{n-1}(z,w)$, where k_{n-1} represents the reproducing kernel for $\mathcal{L}_{n-1}$, that is $k_{n-1}(z,w) = \sum_{i=0}^{n-1}\varphi_i(z)\overline{\varphi_i(w)}$. It reproduces in the sense that $\langle f, k_{n-1}(\cdot,w)\rangle = f(w)$ for any $f \in \mathcal{L}_{n-1}$. This implies, for example, that we only need to know the first n polynomials $\varphi_0, \ldots, \varphi_{n-1}$ to find the n knots $\{\xi_{nk}\}_{k=1}^n$.

Theorem 2.1 *[32] If (2.2) is a Szegő formula, then the distinct knots $\{\xi_{ni}\}_{i=1}^n \subset \mathbb{T}$ are given by the zeros of the para-orthogonal functions $Q_n(z;\tau_n) = \varphi_n(z) + \tau_n\varphi_n^*(z)$ with $\tau_n \in \mathbb{T}$ arbitrary, or equivalently by some arbitrary point $\xi_{n1} \in \mathbb{T}$ and the zeros of $k_{n-1}(z,\xi_{n1})$. The (positive) weights A_{ni} are given by the reciprocals*

$$A_{ni}^{-1} = \sum_{k=0}^{n-1} |\varphi_k(\xi_{ni})|^2 = k_{n-1}(\xi_{ni},\xi_{ni}).$$

To study error formulas and convergence properties, we need the link with moment problems and certain rational approximations.

Therefore, we need to introduce some rational approximants to the Riesz-Herglotz transform of the measure μ. Let us start by defining the *Riesz-Herglotz transform* as

$$F_\mu(z) = I_\mu\{D(\cdot,z)\}, \quad D(t,z) = \frac{t+z}{t-z},$$

which is a function analytic in $\hat{\mathbb{C}} \setminus \mathbb{T}$ having a radial limit a.e. to the unit circle whose real part is the absolutely continuous part of μ. Moreover, it has expansions in $\mathbb{D}$ and $\mathbb{E}$ that can be described in terms of the moments $c_k = I_\mu\{z^{-k}\}$, $k \in \mathbb{Z}$. We have

$$\begin{aligned} F_\mu(z) \sim L_0(z) &= c_0 + 2\sum_{j=1}^{\infty} c_k z^k, \quad z \in \mathbb{D}, \\ F_\mu(z) \sim L_\infty(z) &= -c_0 - 2\sum_{j=1}^{\infty} c_{-k} z^{-k}, \quad z \in \mathbb{E}. \end{aligned}$$

Here we have a motivation for finding approximants that converge to F_μ, since this could help solving the moment problem. Some rational approximants with fixed denominators are constructed as follows.

Consider a triangular table $\mathbb{X} = \{\xi_{ni} \in \mathbb{T} : i = 1, \ldots, n; n \in \mathbb{N}\}$, where $\xi_{ni} \neq \xi_{nj}$ for $i \neq j$. We shall use the rows of this array as knots for quadrature formulas. Therefore, we shall call such an array a *node array*. Let $Q_n \in \Pi_n$ be a node polynomial, that is, a polynomial whose zeros are $\{\xi_{ni}, i = 1, \ldots, n\}$. For any such polynomial Q_n, and for any pair of nonnegative integers (p, q) such that $p + q = n - 1 \geq 0$, we can find a unique polynomial $P_n \in \Pi_n$ such that for $F_{\mu_n} = P_n/Q_n$ we have

$$\begin{aligned} F_\mu(z) - F_{\mu_n}(z) &= O[z^{p+1}], \quad z \to 0 \\ F_\mu(z) - F_{\mu_n}(z) &= O[(1/z)^{q+1}], \quad z \to \infty. \end{aligned}$$

The rational function F_{μ_n} is called a *two-point Padé-type approximation* (2PTA). The relation with quadrature formulas is that if the zeros of Q_n are the abscissas of the n-point Szegő quadrature formula, then the 2PTA $F_{\mu_n} = P_n/Q_n$ has the partial fraction expansion

$$F_{\mu_n}(z) = \sum_{i=1}^{n} A_{ni} D(\xi_{ni}, z),$$

where the A_{ni} are the weights of the quadrature formula. Now, assume that $\mathbb{G}$ is a neighborhood of $\mathbb{T}$ with a boundary $\Gamma = \partial\mathbb{G}$ consisting of finitely many rectifiable curves. Then, by Cauchy's theorem we have for f analytic in the closure of $\mathbb{G}$ and $t \in \mathbb{G}$

$$f(t) = \frac{1}{2\pi i}\int_\Gamma D(t, z) g(z) \mathrm{d}z, \quad g(z) = -f(z)/(2z).$$

Note that if $0 \in \mathbb{G}$ and $f(0) \neq 0$, then we can consider $h(z) = f(z) - f(0)$ since for the error $E_{\mu_n}\{\cdot\} = I_\mu\{\cdot\} - I_{\mu_n}\{\cdot\}$ we have $E_{\mu_n}\{f\} = E_{\mu_n}\{h\}$. Now applying the operator I_μ and using Fubini's theorem, we get

$$I_\mu\{f\} = \frac{1}{2\pi i}\int_\Gamma F_\mu(z) g(z) \mathrm{d}z \quad \text{and} \quad I_{\mu_n}\{f\} = \frac{1}{2\pi i}\int_\Gamma F_{\mu_n}(z) g(z) \mathrm{d}z.$$

Thus, for the error, one has

Theorem 2.2 *[11] Let f be analytic in the closure of $\mathbb{G}$ where $\mathbb{G}$ is a neighborhood of $\mathbb{T}$ whose boundary $\Gamma = \partial\mathbb{G}$ consists of finitely many rectifiable curves. Then, if the triangular table $\mathbb{X}$, the 2PTA $F_{\mu_n} = P_n/Q_n$, and the n-point Szegő formula are as above, we have*

$$E_{\mu_n}\{f\} = I_\mu\{f\} - I_{\mu_n}\{f\} = \frac{1}{2\pi i}\int_\Gamma [F_\mu(t) - F_{\mu_n}(t)] \frac{-f(t)\mathrm{d}t}{2t}$$

and

$$R_{\mu_n}(z) := F_\mu(z) - F_{\mu_n}(z) = \frac{2z^{p+1}}{Q_n(z)}\int_{-\pi}^{\pi} \frac{Q_n(e^{i\theta}) e^{-ip\theta}}{e^{i\theta} - z} \mathrm{d}\mu(\theta).$$

This shows that there is an intimate relationship between the convergents of 2PTA for F_μ and the convergence of quadrature formulas.

We shall not develop our treatment of the polynomial case any further, but use the thread of this section as a motivation for the more general case of rational functions replacing the polynomials. We shall do this in the next sections. The polynomial situation is there just a special case.

3 Rational Szegő formulas and modified approximants

Let $\mathbb{A} = \{\alpha_n : n = 1, 2, \ldots\}$ be an arbitrary sequence of points in $\mathbb{D}$. Define the Blaschke factors ζ_k by

$$\zeta_k(z) = \frac{\overline{\alpha}_k}{|\alpha_k|}\frac{\alpha_k - z}{1 - \overline{\alpha}_k z}, \quad k = 1, 2, \ldots, \quad \text{where } \frac{\overline{\alpha}_k}{|\alpha_k|} = -1 \text{ if } \alpha_k = 0,$$

and the Blaschke products $B_0 = 1$ and $B_k = \zeta_1 \cdots \zeta_k$, $k \geq 1$. The spaces of polynomials of the previous section are replaced by the spaces of rational functions $\mathcal{L}_n = \operatorname{span}\{B_0, \ldots, B_n\}$. Note that if we set $\pi_0 = 1$ and $\pi_n(z) = \prod_{k=1}^n (1 - \overline{\alpha}_k z)$, $n \geq 1$, then $f \in \mathcal{L}_n$ is of the form p/π_n with $p \in \Pi_n$. The spaces of negative powers of z are replaced by $\mathcal{L}_{n*} = \operatorname{span}\{B_0, B_{1*}, \ldots, B_{n*}\}$. Thus, setting $\omega_0 = 1$ and $\omega_n(z) = \prod_{k=1}^n (z - \alpha_k)$, $n \geq 1$, then $f \in \mathcal{L}_{n*}$ is of the form q/ω_n with $q \in \Pi_n$. The space of L-polynomials is replaced by $\mathcal{R}_{p,q} = \mathcal{L}_{p*} + \mathcal{L}_q = \{N/(\pi_q \omega_p) : N \in \Pi_{p+q}\}$. Note that if all $\alpha_k = 0$, then we are back in the situation of the polynomials and the L-polynomials as in the previous section.

Let $\hat{\mathbb{A}} = \{1/\overline{\alpha}_k : \alpha_k \in \mathbb{A}\}$. Since $\mathcal{R}_{p,q}$ is a Chebyshev system on any set $\mathbb{X} \subset \mathbb{C}\setminus(\mathbb{A} \cup \hat{\mathbb{A}})$, it follows that for any $i = 1, \ldots, n$, there is a unique rational function $L_{ni} \in \mathcal{R}_{p,q}$, $p+q = n-1 \geq 0$, such that it satisfies $L_{ni}(\xi_{nj}) = \delta_{ij}$, where as before $\mathbb{X} = \{\xi_{ni}, i = 1, \ldots, n; n = 1, 2, \ldots\}$ is a triangular table of points on $\mathbb{T}$ such that $\xi_{ni} \neq \xi_{nj}$ for $i \neq j$. Hence $f_n(z) = \sum_{i=1}^n L_{ni}(z) f(\xi_{ni})$ is the unique function from $\mathcal{R}_{p,q}$ interpolating a given function f in the points ξ_{ni}, $i = 1, \ldots, n$, and $I_{\mu_n}\{f\} := I_\mu\{f_n\} = \sum_{i=1}^n A_{ni} f(\xi_{ni})$ with $A_{ni} = I_\mu\{L_{ni}\}$ is a *quadrature formula of interpolatory type* having *domain of validity* $\mathcal{R}_{p,q}$.

Again, by an appropriate choice of the knots ξ_{ni}, we want to extend the domain of validity to make it as large as possible. As in the L-polynomial case, it can be shown that

Theorem 3.1 *[9] There does not exist an n-point quadrature formula of the form (2.2) with distinct knots on the unit circle that is exact in $\mathcal{R}_{n-1,n}$ or $\mathcal{R}_{n,n-1}$.*

This means that $\mathcal{R}_{n-1,n-1}$ is a candidate for a *maximal domain of validity.* As in the polynomial case, we can obtain this maximal domain of validity by choosing the abscissas as the zeros of the para-orthogonal functions. Such an optimal formula is called a *rational Szegő formula* (or R-Szegő formula for short).

Therefore we have to extend our previous notion of para-orthogonality. First define the operation indicated by a superstar. For any $f_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ ($\mathcal{L}_{-1} = \varnothing$), we set $f_n^* = B_n f_{n*}$. This generalizes the superstar conjugate for polynomials, since indeed if all α_k are zero these notions coincide. Suppose that by Gram-Schmidt orthogonalization of the Blaschke products $\{B_n\}$, we generate an orthogonal sequence $\{\varphi_n\}$. Then $0 = \langle \mathcal{L}_{n-1}, \varphi_n \rangle = \langle \phi_{n*}, \mathcal{L}_{(n-1)*} \rangle = \langle \varphi_n^*, B_n \mathcal{L}_{(n-1)*} \rangle$. Now note that $B_n \mathcal{L}_{(n-1)*} = \{f \in \mathcal{L}_n : f(\alpha_n) = 0\}$. Thus, if we set $\mathcal{L}_n(w) = \{f \in \mathcal{L}_n : f(w) = 0\}$, then $B_n \mathcal{L}_{(n-1)*} = \mathcal{L}_n(\alpha_n)$. Thus, $\varphi_n \perp \mathcal{L}_{n-1} \Leftrightarrow \varphi_n^* \perp \mathcal{L}_n(\alpha_n)$. Moreover, note that $\langle \varphi_n, B_n \rangle = \langle 1, \varphi_n^* \rangle \neq 0$. This motivates the following definition.

Definition 3.1 *We say that a sequence of functions $Q_n \in \mathcal{L}_n$ is para-orthogonal if $Q_n \perp \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n)$ for $n \geq 1$ while $\langle Q_n, 1 \rangle \neq 0$ and $\langle Q_n, B_n \rangle \neq 0$.*

Definition 3.2 *A function $Q_n \in \mathcal{L}_n$ is called c-invariant if $Q_n^* = cQ_n$ for some nonzero constant $c \in \mathbb{C}$.*

It can be shown that any para-orthogonal and k_n-invariant function $Q_n \in \mathcal{L}_n$ has to be some constant multiple of $Q_n(z; \tau_n) = \varphi_n(z) + \tau_n \varphi_n^*(z)$ with $\tau_n \in \mathbb{T}$. The most important property is stated in the next theorem.

Theorem 3.2 *[9] Any para-orthogonal and k_n-invariant function from $\mathcal{L}_n$ has precisely n simple zeros on $\mathbb{T}$.*

Thus the functions $Q_n(z; \tau_n) = \varphi_n(z) + \tau_n \varphi_n^*(z)$ with $\tau_n \in \mathbb{T}$ can provide the knots for an R-Szegő formula and indeed they do, and what is more: this is the only possibility.

Theorem 3.3 *[9] The quadrature formula (2.2) with distinct knots on $\mathbb{T}$ is an R-Szegő formula (with maximal domain of validity $\mathcal{R}_{n-1,n-1}$) if and only if*
(a) it is of interpolatory type in $\mathcal{R}_{p,q}$ with p, q nonnegative integers with $p + q = n - 1$;
(b) the abscissas are the zeros of a para-orthogonal k_n-invariant function from $\mathcal{L}_n$.

Note that for each n, we have a one-parameter family of R-Szegő quadrature formulas, since the parameter $\tau_n \in \mathbb{T}$ is free.

Let us now introduce the reproducing kernels, since both the abscissas and the weights can be expressed in terms of these kernels. If $\{\varphi_k\}$ are the orthonormal functions, then the kernel function $k_n(z, w) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(w)}$ is reproducing for $\mathcal{L}_n$, meaning that $\langle f, k_n(\cdot, w) \rangle = f(w)$ for all $f \in \mathcal{L}_n$. Like for the Szegő polynomials, these kernels appear in a Christoffel-Darboux formula.

Theorem 3.4 *[8] Let $\{\varphi_k\}$ be the orthonormal polynomials for the spaces $\mathcal{L}_n$. Then the reproducing kernel satisfies*

$$k_{n-1}(z, w) = \frac{\varphi_n^*(z)\overline{\varphi_n^*(w)} - \varphi_n(z)\overline{\varphi_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}}.$$

So far we have characterized the abscissas of the R-Szegő formulas as the zeros of the para-orthogonal functions $Q_n(z; \tau_n)$. They can also be written as the zeros of a reproducing kernel. Indeed, by varying $\tau_n \in \mathbb{T}$, we can place for example ξ_{n1} at any position $w \in \mathbb{T}$. The other $n - 1$ zeros ξ_{ni} are then the zeros of $k_{n-1}(z, w)$ and conversely: choose $\xi_{n1} \in \mathbb{T}$ arbitrary and let $\{\xi_{ni}\}_{i=2}^{n}$ be the zeros of $k_{n-1}(z, \xi_{n1})$, then there is some $\tau_n \in \mathbb{T}$ such that these $\{\xi_{ni}\}_{i=1}^{n}$ are the zeros of the para-orthogonal function $Q_n(z; \tau_n)$ (see [7]).

Also the weights can be expressed in terms of the kernels exactly like in the polynomial case. This results in the fact that Theorem 2.1 is still true if we replace Szegő formula by R-Szegő formula (see [13]).

The 2PTA of the previous section are generalized to *multipoint rational approximants* (MRA) in the following sense. Let $Q_n = \varphi_n + \tau_n \varphi_n^*$, $\tau_n \in \mathbb{T}$, be the para-orthogonal function as above. Now define the so-called functions of the second kind $\psi_n \in \mathcal{L}_n$ as

$$\psi_n(z) = I_\mu\{E(t, z)\varphi_n(t) - D(t, z)\varphi_n(z)\}, \quad E(t, z) = D(t, z) + 1, \tag{3.1}$$

and set $P_n = \psi_n - \tau_n\psi_n^*$. Then it turns out that the rational function $F_{\mu_n}(\cdot;\tau_n) := F_{\mu_n} = -P_n/Q_n$ (depending on z and τ_n) is a MRA for the Riesz-Herglotz transform F_μ, since

$$zB_{n-1}(z)[F_\mu(z) - F_{\mu_n}(z)] \quad \text{and} \quad [zB_{n-1}(z)]_*[F_\mu(z) - F_{\mu_n}(z)]$$

are both analytic in $\hat{\mathbb{C}}\backslash\mathbb{T}$. This means that F_{μ_n} interpolates F_μ in the points $\{0, \alpha_1, \ldots, \alpha_{n-1}\}$ and in $\{\infty, 1/\overline{\alpha}_1 \ldots, 1/\overline{\alpha}_{n-1}\}$. Note that there are $2n+1$ degrees of freedom while there are $2n$ interpolation conditions. So there is one condition short for F_{μ_n} to be a multipoint Padé approximant. These approximants are called *modified approximants* (MA) since they are modifications of the true multipoint Padé approximants (MPA) $F_{\mu_n}(\cdot;0) = \varphi_n/\psi_n$ (interpolates in all the MA points and in the extra point $1/\overline{\alpha}_n$) and $F_{\mu_n}(\cdot;\infty) = -\psi_n^*/\varphi_n^*$ (interpolates in all the MA points and in the extra point α_n). Furthermore, it follows from the partial fraction expansion $F_{\mu_n}(z) = \sum_{i=1}^n A_{ni}D(\xi_{ni}, z)$ that $F_{\mu_n}(z) = I_{\mu_n}\{D(\cdot, z)\}$, and this relates it to the quadrature formula.

If more generally, we have a rational function F_{μ_n} whose denominator is a node polynomial of degree n for some node array, and suppose it interpolates F_μ in the points $\{0, \alpha_1, \ldots, \alpha_p\}$ and in $\{\infty, 1/\overline{\alpha}_1 \ldots, 1/\overline{\alpha}_q\}$, then we say that it is an MRA of order $(p+1, q+1)$. Thus, our MA is an MRA of order (n, n). Let $F_{\mu_n}(z) = \sum_{i=1}^n A_{ni}D(\xi_{ni}, z)$ be the partial fraction decomposition of F_{μ_n} which defines the weights A_{nj} as

$$A_{nj} = \frac{\omega_p(\xi_{nj})\pi_q(\xi_{nj})}{X_n'(\xi_{nj})} I_\mu\left\{\frac{X_n(t)}{\omega_p(t)\pi_q(t)(t-\xi_{nj})}\right\}, \quad X_n(t) = \prod_{i=1}^n (t - \xi_{ni});$$

then it can be shown that for given $(\mathbb{A}, \mathbb{X})$, the quadrature formula $I_{\mu_n}\{f\} = \sum_{i=1}^n A_{ni}f(\xi_{ni})$ is exact in $\mathcal{R}_{p,q}$ if and only if F_{μ_n} is an MRA of type $(p+1, q+1)$ with respect to the point sets $(\mathbb{A}, \mathbb{X})$ for the Riesz-Herglotz transform F_μ (see [15]).

We can now work towards an expression for the error of the quadrature formula. Assume f is analytic in a region $\mathbb{G} \subset \mathbb{C} \setminus (\mathbb{A} \cup \hat{\mathbb{A}})$ that includes $\mathbb{T}$. This is only possible if $\mathbb{A}$ is in a compact subset of $\mathbb{D}$, i.e., if the α_k do not tend to $\mathbb{T}$. Exactly the same type of proof as in the polynomial case can be given for the following theorem.

Theorem 3.5 *[12] Let f be analytic in a closed region $\mathbb{G}$ such that $\mathbb{T} \subset \mathbb{G} \subset \mathbb{C} \setminus (\mathbb{A} \cup \hat{\mathbb{A}})$ with a boundary $\Gamma = \partial\mathbb{G}$ that consists of finitely many rectifiable curves. Let $\mathbb{X}$ be a given node array. Assume that $I_{\mu_n}\{f\} = \sum_{i=1}^n A_{ni}f(\xi_{ni})$ is exact in $\mathcal{R}_{p,q}$ and hence $F_{\mu_n} = \sum_{i=1}^n A_{ni}D(\xi_{ni}, \cdot)$ is an MRA of order $(p+1, q+1)$ for F_μ in the strong sense; then*

$$E_{\mu_n}\{f\} := I_\mu\{f\} - I_{\mu_n}\{f\} = \frac{1}{2\pi i}\int_\Gamma [F_\mu(t) - F_{\mu_n}(t)]\frac{-f(t)\mathrm{d}t}{2t}. \tag{3.2}$$

Define the node polynomial $X_n(z) = \prod_{i=1}^n (z - \xi_{ni})$; then

$$R_{\mu_n}(z) := F_\mu(z) - F_{\mu_n}(z) = \frac{2z\omega_p(z)\pi_q(z)}{X_n(z)}\int_{-\pi}^{\pi} \frac{X_n(e^{i\theta})}{(e^{i\theta} - z)\omega_p(e^{i\theta})\pi_q(e^{i\theta})}\mathrm{d}\mu(\theta). \tag{3.3}$$

There is no MRA of degree n with simple poles in $\mathbb{T}$ and of order $(n+1, n)$ or order $(n, n+1)$, hence there is no quadrature formula with knots on $\mathbb{T}$ with domain of validity $\mathcal{R}_{n,n-1}$ or $\mathcal{R}_{n-1,n}$.

The only quadrature formulas exact in $\mathcal{R}_{n-1,n-1}$ with knots on $\mathbb{T}$ are the R-Szegő formulas, and hence the MA are the only MRA of order (n, n), i.e., the ones whose poles are zeros of the para-orthogonal function of degree n.

Here too, it is seen that the convergence of the quadrature formulas is closely related to the convergence of the MA's or MRA's.

By means of orthogonality properties it can be shown that for the MA's the previous error formula can be transformed into

$$R_{\mu_n}(z) = \frac{2z\omega_{n-1}(z)\pi_{n-1}(z)}{[X_n(z)]^2}\left[\int_{-\pi}^{\pi}\frac{[X_n(e^{i\theta})]^2}{(e^{i\theta}-z)\omega_{n-1}(e^{i\theta})\pi_{n-1}(e^{i\theta})}\mathrm{d}\mu(\theta)+\delta_n\right], \tag{3.4}$$

where $\delta_n = I_\mu\{Q_n(z)(1-\overline{\alpha}_n z)\}$. Note that this term δ_n is caused by the fact that para-orthogonality is a defective orthogonality. When in classical formulas, zeros of orthogonal polynomials are used, then such a term does not appear.

4 Orthogonal rational functions

The quadrature formulas we consider in this paper are closely related to the properties of orthogonal and quasi-orthogonal rational functions. We collect some properties of these functions in this section.

The orthonormal rational functions $\{\varphi_0, \varphi_1, \ldots\}$ are obtained by orthonormalization of the sequence of Blaschke products $\{B_0, B_1, \ldots\}$. They were first considered by Djrbashian (see the references in [23]). Later on, the reproducing kernels were considered in the context of linear prediction and the Nevanlinna-Pick interpolation problem [4]. We assume that they are normalized by the condition that in the expansion $\varphi_n(z) = \sum_{k=0}^n a_{nk}B_k(z)$, the leading coefficient with respect to this basis is positive: $\kappa_n = a_{nn} > 0$. If $k_n(z,w) = \sum_{k=0}^n \varphi_k(z)\overline{\varphi_k(w)}$ is the reproducing kernel for $\mathcal{L}_n = \operatorname{span}\{B_0,\ldots,B_n\}$, then it can be verified that $k_n(z,\alpha_n) = \kappa_n\varphi_n^*(z)$, where $\varphi_n^*(z) = B_n(z)\varphi_{n*}(z)$, hence also $k_n(\alpha_n,\alpha_n) = \kappa_n^2$. Although it is not essential for the present application, we mention for completeness that the φ_n satisfy a recurrence relation, i.e., there exist specific constants ε_n and δ_n such that $|\varepsilon_n|^2 - |\delta_n|^2 = \frac{\kappa_{n-1}^2(1-|\alpha_n|^2)}{\kappa_n^2(1-|\alpha_{n-1}|^2)} > 0$ and

$$\varphi_n(z) = \frac{\kappa_n}{\kappa_{n-1}}\left[\varepsilon_n\frac{z-\alpha_{n-1}}{1-\overline{\alpha}_n z}\varphi_{n-1}(z) + \delta_n\frac{1-\overline{\alpha}_{n-1}z}{1-\overline{\alpha}_n z}\varphi_{n-1}^*(z)\right].$$

The initial condition is $\varphi_0 = 1/\sqrt{c_0}$ with $c_0 = \int_{\mathbb{T}}\mathrm{d}\mu$. There is also a Favard type theorem: if some φ_n satisfy a recurrence relation of this form, then they will form an orthonormal sequence with respect to some positive measure on $\mathbb{T}$. In this respect see also the contribution by Marcellán and Alvarez in this volume.

The functions of the second kind ψ_n associated with φ_n are another independent solution of the same recurrence relation. They can also be derived from the φ_n by the relation (3.1). In fact, this means that $\psi_0 = \sqrt{c_0} = 1/\varphi_0$ and $\psi_n = I_\mu\{D(\cdot,z)[\varphi_n(\cdot) - \varphi(z)]\}$ for $n \geq 1$. The para-orthogonal functions $Q_n(z;\tau_n) = \varphi_n(z) + \tau_n\varphi_n^*(z)$ and the associated functions of the second kind $P_n(z;\tau_n) = \psi_n(z) - \tau_n\psi_n^*(z)$ were introduced before.

Example 4.1 **[Malmquist basis]** Assume we take the normalized Lebesgue measure $\mathrm{d}\mu(\theta) = \mathrm{d}\theta/(2\pi)$. Then it is known that the orthonormal functions are given by

$$\varphi_n(z) = \kappa_n\frac{zB_n(z)}{z-\alpha_n}, \qquad \kappa_n = \sqrt{1-|\alpha_n|^2}.$$

This basis is known as the Malmquist basis. Then $\varphi_n^*(z) = \kappa_n/(1-\overline{\alpha_n}z)$ and therefore $Q_n(z;\tau_n) = \varphi_n(z) + \tau_n\varphi_n^*(z) = \kappa_n[\frac{zB_n(z)}{z-\alpha_n} + \frac{\tau_n}{1-\overline{\alpha}_n z}]$. Noting that $B_n(z) = \eta_n\omega_n(z)/\pi_n(z)$, with

$\eta_n \in \mathbb{T}$, and $\tau_n \in \mathbb{T}$ is arbitrary, we can choose $\tau_n = \eta_n$, so that the expression for $Q_n(z;\tau_n)$ becomes $Q_n(z;\tau_n) = \kappa_n\tau_n[z\omega_{n-1}(z)+\pi_{n-1}(z)]/\pi_n(z)$. If all $\alpha_k = 0$, then $\omega_{n-1}(z) = z^{n-1}$ and $\pi_{n-1}(z) = 1$, so that the zeros of $Q_n(z;\tau_n)$ are (a rotated version of) the nth roots of unity.

We also have to introduce the spaces $\mathcal{R}_{p,q} = \text{span}\{B_{-p},\ldots,B_q\}$, where p,q are nonnegative integers and $B_{-p} = 1/B_p = B_{p*}$. We set $\mathcal{L} = \bigcup_{n=0}^{\infty}\mathcal{L}_n$ and $\mathcal{R} = \bigcup_{n=0}^{\infty}\mathcal{R}_{n,n}$.

Theorem 4.1 *[12] The space $\mathcal{L}$ is dense in $H^p(\mathbb{D})$, $1 \le p < \infty$, if and only if $\sum_k(1-|\alpha_k|) = \infty$.*
The space $\mathcal{R}$ is dense in $L^p(\mathbb{T})$, $1 \le p < \infty$, and in $C(\mathbb{T})$ if and only if $\sum_k(1-|\alpha_k|) = \infty$.

We note that the condition $\sum_k(1-|\alpha_k|) = \infty$ means that the Blaschke product B_n converges uniformly to zero in $\mathbb{D}$. Also we should have $p \ne \infty$ in this theorem, and not $1 \le p \le \infty$ as erroneously stated in [12].

Finally, we mention the rational variant of the trigonometric moment problem. Suppose a linear functional M is defined on $\mathcal{L}$ by the moments

$$c_0 = M\{1\}, \quad c_k = M\{1/\pi_k\}, \quad \pi_k(z) = \prod_{i=1}^{k}(1-\overline{\alpha}_k z), \quad k = 1,2,\ldots$$

and by $M\{\omega_{k*}\} = \overline{c}_k$; we can then define M on the whole of $\mathcal{R} = \mathcal{L}+\mathcal{L}_* = \mathcal{L}\cdot\mathcal{L}_*$, where $\mathcal{L}_* = \{f_* : f \in \mathcal{L}\}$. The moment problem is to find under what conditions there exists a positive measure μ on $\mathbb{T}$ such that $M\{\cdot\} = I_\mu\{\cdot\}$, and if the problem is solvable, to find conditions under which the solution is unique and possibly, if there are more solutions, to describe all of them. For a proof of the following results about moment problems, we refer to [14, 19].

If it is assumed that M satisfies $M\{f_*\} = \overline{M\{f\}}$ for all $f \in \mathcal{R}$ and $M\{ff_*\} > 0$ for all nonzero $f \in \mathcal{L}$, then this functional defines an inner product $\langle f,g\rangle_M = M\{fg_*\}$. Under these conditions one can guarantee that a solution for the moment problem exists. Denote by $\mathcal{M}$ the set of all solutions. For solving the uniqueness question, the MRA's that we considered in the previous section play a central role in the solution of the problem. Recall that the MRA is given by $F_n(z;\tau_n) = -P_n(z;\tau_n)/Q_n(z;\tau_n)$, $\tau_n \in \mathbb{T}$, where $Q_n(z;\tau_n)$ are the para-orthogonal functions and $P_n(z;\tau_n)$ the associated functions of the second kind. It turns out that the set $K_n(z) = \{F_n(z;\tau) : \tau \in \mathbb{T}\}$ is a circle. Moreover, the circular disks: $\Delta_n(z)$ with boundary $K_n(z)$ are nested: $\Delta_{n+1}(z) \subset \Delta_n(z)$ and their boundaries touch. The limiting set $\Delta(z) = \bigcap_n \Delta_n(z)$ will be either one point or a circular disk, and this fact is independent of the value chosen for the complex number $z \in \mathbb{C}\setminus(\mathbb{A}\cup\hat{\mathbb{A}})$. If the Blaschke product diverges, i.e., $\sum(1-|\alpha_k|) = \infty$, then the limiting set is a point and the moment problem has a unique solution (is determinate). If the limiting set $\Delta(z)$ is a disk with positive radius, then the Blaschke product converges, i.e., $\sum(1-|\alpha_k|) < \infty$, and the moment problem has infinitely many solutions. The set $\Delta(z)$ can be characterized as $\Delta(z) = \{F_\mu(z) : \mu \in \mathcal{M}\}$, where F_μ denotes the Riesz-Herglotz transform of μ. A solution $\mu \in \mathcal{M}$ is called N-extremal if its Riesz-Herglotz transform F_μ belongs to the boundary of $\Delta(z)$. It can be proved that $\mu \in \mathcal{M}$ is N-extremal if and only if $\mathcal{L}$ is dense in L^2_μ.

The last density result is interesting because N-extremal solutions exist if the Blaschke product converges. Thus, $\mathcal{L}$ may be dense in L^2_μ even if $\sum(1-|\alpha_k|) < \infty$. However, if the Blaschke product diverges, then $\mathcal{L}$ will be dense in L^2_μ.

5 Convergence of MA and R-Szegő quadrature

We are still considering the case of a positive measure μ and MA's $F_{\mu_n} = P_n/Q_n$, where $Q_n = \varphi_n + \tau_n \varphi_n^*$, $\tau_n \in \mathbb{T}$, is the para-orthogonal function in $\mathcal{L}_n$ and $P_n = \psi_n - \tau_n \psi_n^*$ is its associated function. Also $I_{\mu_n}\{f\}$ is the nth R-Szegő formula. We have seen that the convergence of $I_{\mu_n}\{f\}$ is related to the convergence of F_{μ_n}. That F_{μ_n} does converge is essentially a consequence of the Stieltjes-Vitali theorem.

Theorem 5.1 *[8, 15] If $\sum_{n=1}^{\infty}(1-|\alpha_n|) = \infty$, then the MA's $F_{\mu_n}(z;\tau_n)$ converge to $F_\mu(z)$ uniformly on compact subsets of $\hat{\mathbb{C}} \setminus \mathbb{T}$.*

To estimate the rate of convergence, we recall that $Q_n = X_n/\pi_n$ and $|B_n| = |\omega_n/\pi_n|$, so that from (3.4)

$$|F_{\mu_n}(z)| \le 2|z| \left|\frac{\omega_{n-1}(z)}{\pi_n(z)}\right| \left|\frac{\pi_{n-1}(z)}{\pi_n(z)}\right| \frac{1}{|Q_n(z)|^2} \left[I_\mu \left\{ \left|\frac{\pi_n}{\pi_{n-1}}\right| \left|\frac{\pi_n}{\omega_{n-1}}\right| \frac{|Q_n|^2}{|\cdot - z|} \right\} + |\delta_n| \right],$$

and hence there is a constant M not depending on n such that

$$|F_{\mu_n}(z)|^{1/n} \le M^{1/n} [|B_{n-1}(z)||1-\overline{\alpha}_n z|^2 |Q_n(z)|^2]^{-1/n} (S_n + |\delta_n|)^{1/n},$$

with $S_n = \|Q_n^2\|_\infty \max_{t\in\mathbb{T}} |1-\overline{\alpha}_n t|^2$. This explains why we need the root asymptotics of Q_n and B_n to estimate the rate of convergence for F_{μ_n}. Therefore we need some assumptions on μ and the point set $\mathbb{A} = \{\alpha_1, \alpha_2, \ldots\}$. For the given set $\mathbb{A}$, let $\nu_n^{\mathbb{A}} = \frac{1}{n}\sum_{j=1}^n \delta_{\alpha_j}$ be the counting measure, which assigns a mass at α_j proportional to its multiplicity. Assume that $\nu_n^{\mathbb{A}}$ converges to some $\nu^{\mathbb{A}}$ in the weak star sense, which we denote as $\nu_n^{\mathbb{A}} \xrightarrow{*} \nu^{\mathbb{A}}$. Then the root asymptotics for the Blaschke products are given by

Theorem 5.2 *[15] If B_n is the Blaschke product with zeros $\{\alpha_k\}_{k=1}^n$, and $\nu_n^{\mathbb{A}} \xrightarrow{*} \nu^{\mathbb{A}}$, then*

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = \exp\{\lambda(z)\} \quad \textit{and} \quad \lim_{n\to\infty} |B_n(z)|^{-1/n} = \exp\{\lambda(\hat{z})\}$$

locally uniformly for $z \in \mathbb{C} \setminus \left(\{0\} \cup \operatorname{supp}(\nu^{\mathbb{A}}) \cup \operatorname{supp}(\nu^{\hat{\mathbb{A}}})\right)$, where

$$\lambda(z) = \int \log|\zeta_z(x)| d\nu^{\mathbb{A}}(x), \quad \zeta_z(x) = \frac{x-z}{1-\overline{z}x}, \tag{5.1}$$

and where $\hat{z} = 1/\overline{z}$ and $\hat{\mathbb{A}} = \{\hat{\alpha} = 1/\overline{\alpha} : \alpha \in \mathbb{A}\}$. For $z \in \mathbb{C} \setminus \{0\}$ we have the inequalities

$$\limsup_{n\to\infty} |B_n(z)|^{1/n} \le \exp\{\lambda(z)\} \quad \textit{and} \quad \limsup_{n\to\infty} |B_n(z)|^{-1/n} \le \exp\{\lambda(\hat{z})\}.$$

As for the root asymptotics of the para-orthogonal functions, one has

Theorem 5.3 *[15] Let μ be a positive measure satisfying the Szegő condition $\int_{-\pi}^{\pi} \log \mu'(\theta) \mathrm{d}\theta > -\infty$ and assume that the point set $\mathbb{A}$ is compactly included in $\mathbb{D}$ and that $\nu_n^{\mathbb{A}} \xrightarrow{*} \nu^{\mathbb{A}}$. Then, for the para-orthogonal functions Q_n, it holds locally uniformly in the indicated regions that*

$$\begin{aligned}
&\lim_{n\to\infty} |Q_n(z)|^{1/n} = 1, \quad z \in \mathbb{D},\\
&\lim_{n\to\infty} |Q_n(z)|^{1/n} = \exp\{\lambda(z)\}, \quad z \in \mathbb{E} \setminus \operatorname{supp}(\nu^{\hat{\mathbb{A}}}),\\
&\limsup_{n\to\infty} |Q_n(z)|^{1/n} \le \exp\{\lambda(z)\}, \quad z \in \mathbb{E},\\
&\lim_{n\to\infty} \|Q_n(z)\|_\infty^{1/n} = 1.
\end{aligned}$$

A combination of the previous results now leads to the rate of convergence for the MA's.

Theorem 5.4 *[15] Under the same conditions as in the previous theorem, the following estimates hold for the convergence of the MA's F_{μ_n} to the Riesz-Herglotz transform F_μ. Setting $R_{\mu_n} = F_\mu - F_{\mu_n}$, then*
$\limsup_{n\to\infty} |R_{\mu_n}(z)|^{1/n} \le \exp\{\lambda(z)\} < 1$, $\forall z \in \mathbb{D}$,
$\limsup_{n\to\infty} |R_{\mu_n}(z)|^{1/n} \le \exp\{\lambda(\hat{z})\} < 1$, $\forall z \in \mathbb{E}$, *where* $\hat{z} = 1/\overline{z}$,
and $\lambda(z)$ *as in (5.1).*

Example 5.1 Consider the simple case where $\lim_{n\to\infty} \alpha_k = a \in \mathbb{D}$. Then $\nu^{\mathbb{A}}(z) = \delta_a$ and $\lambda(z) = \log|\zeta_z(a)|$, $\zeta_z(a) = (a-z)/(1-\overline{z}a)$. Therefore, $\limsup_{n\to\infty} |R_{\mu_n}(z)|^{1/n} \le |\zeta_z(a)| < 1$ for $z \in \mathbb{D}$ and $\limsup_{n\to\infty} |R_{\mu_n}(z)|^{1/n} \le 1/|\zeta_z(a)| < 1$ for $z \in \mathbb{E}$. The best rates of convergence are obtained for z near a and $\hat{a} = 1/\overline{a}$, as one could obviously expect.

Similar results hold for the true MPA's:

Theorem 5.5 *[15] Under the same conditions as in the previous theorem, the following estimates hold for the convergence of the MPA's $F_n = \psi_n/\varphi_n$ and $F_n^\times = -\psi_n^*/\varphi_n^*$ to the Riesz-Herglotz transform F_μ. Set $R_n = F_\mu - F_n$ and $R_n^\times = F_\mu - F_n^\times$; then locally uniformly in the indicated regions:*
$\limsup_{n\to\infty} |R_n^\times(z)|^{1/n} \le \exp\{\lambda(z)\} < 1$, $\forall z \in \mathbb{D}$,
$\limsup_{n\to\infty} |R_{\mu_n}(z)|^{1/n} \le \exp\{\lambda(\hat{z})\} < 1$, $\forall z \in \mathbb{E}$, *where* $\hat{z} = 1/\overline{z}$,
and $\lambda(z)$ *as in (5.1).*

Now we can move on to the convergence of the R-Szegő formulas. This is a direct consequence of the previous analysis. For example, we get from (3.2) that

$$|E_{\mu_n}\{f\}| \le \frac{1}{4\pi} \max_{t\in\Gamma} \left|\frac{f(t)}{t}\right| \int_\Gamma |F_\mu(t) - F_{\mu_n}(t)||\mathrm{d}t|.$$

Therefore, it follows under the conditions of Theorems 3.5 and 5.4 that the R-Szegő quadrature formula converges to the integral for all functions analytic in $\mathbb{G}$ with the region $\mathbb{G}$ as above in Theorem 3.5. For this situation, we can even obtain an estimate for the rate of convergence that relies on the previous estimates.

Theorem 5.6 *[15] Let $I_{\mu_n}\{f\}$ be the R-Szegő formula for a function f that is analytic in a closed region $\mathbb{G}$ such that $\mathbb{T} \subset \mathbb{G} \subset \mathbb{C} \setminus (\mathbb{A} \cup \hat{\mathbb{A}})$ whose boundary Γ consists of finitely many rectifiable curves. Then under the conditions of Theorem 5.4,*

$$\limsup_{n\to\infty} |I_\mu\{f\} - I_{\mu_n}\{f\}|^{1/n} \le \gamma < 1,$$

where $\gamma = \max\{\gamma_1, \gamma_2\}$, *with* $\gamma_1 = \max_{z\in\Gamma\cap\mathbb{D}} \exp\{\lambda(z)\}$ *and* $\gamma_2 = \max_{z\in\Gamma\cap\mathbb{E}} \exp\{\lambda(\hat{z})\}$, *where* $\hat{z} = 1/\overline{z}$ *and* $\lambda(z)$ *as in (5.1).*

To prove convergence for the broader class of continuous functions $f \in C(\mathbb{T})$, we define $\gamma_n(f) = \inf_{f_n\in\mathcal{R}_{n,n}} \|f - f_n\|_\infty$. By Theorem 4.1, $\lim_{n\to\infty} \gamma_n(f) = 0$ if $\sum_k (1 - |\alpha_k|) = \infty$. Assume that $r_{n-1} \in \mathcal{R}_{n-1,n-1}$ is such that $\|f - r_{n-1}\|_\infty = \gamma_{n-1}(f)$. If we take into account that $I_{\mu_n}\{r_{n-1}\} = I_\mu\{r_{n-1}\}$, then it follows that

$$|E_{\mu_n}\{f\}| = |I_\mu\{f - r_n\} + I_{\mu_n}\{r_{n-1} - f\}| \le \gamma_{n-1}(f)[I_\mu\{1\} + I_{\mu_n}\{1\}].$$

Thus, $|E_{\mu_n}\{f\}| \leq C_1\gamma_{n-1}(f)$, with C_1 a constant. So it follows from the convergence of γ_n that also the R-Szegő formula converges for continuous functions. If we then also take into account the standard argument proving that, if a quadrature formula with positive coefficients converges for continuous functions, then it also converges for bounded integrable functions, we arrive at

Theorem 5.7 *The R-Szegő formulas $I_{\mu_n}\{f\}$ converge for any bounded integrable function f and positive μ if $\sum_n(1-|\alpha_n|)=\infty$.*

Thus, we have obtained convergence in the largest possible class.

With the help of [12, Theorem 4.7], it can be shown that γ_n can be bounded in terms of the modulus of continuity $\omega(f,\delta) = \sup\{|f(t)-f(\tau)| : t,\tau \in \mathbb{T}, |\mathrm{Arg}(t/\tau)| < \delta\}$. So, there exists a constant C_2 such that $|E_{\mu_n}\{f\}| \leq C_2\omega(f,\pi/(n+1))$ for n large enough.

6 The case of a complex measure

If the measure μ is complex, then we cannot guarantee the existence of a sequence of orthogonal rational functions. In that case we can choose an arbitrary auxiliary positive measure ν on $\mathbb{T}$ and compute the knots of the quadrature formula as the zeros of a para-orthogonal function for this measure. The obvious question is what would be a good choice for this auxiliary measure. Choosing the Lebesgue measure as ν would lead to equidistant nodes on $\mathbb{T}$. There are few other examples of measures that lead to explicit expressions for the knots. In general, they should be computed numerically. If we are prepared to do this, then we could choose the measure ν as a function of the convergence behavior of the quadrature formulas.

In this case we shall consider absolutely continuous measures. So, let $\mathrm{d}\mu(\theta) = \rho(\theta)\mathrm{d}\theta$ and $\mathrm{d}\nu(\theta) = \omega(\theta)\mathrm{d}\theta$. We assume $\omega(\theta) > 0$, $\int_{-\pi}^{\pi} |\rho(\theta)|\mathrm{d}\theta < \infty$, and $\rho/\omega \in L^2_\omega$, i.e.,

$$\int_{-\pi}^{\pi} \frac{|\rho(\theta)|^2}{\omega(\theta)}\mathrm{d}\theta = K^2 < \infty. \tag{6.1}$$

We are concerned with the computation of integrals of the form $I_\rho\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})\rho(\theta)\mathrm{d}\theta$ approximated by $I_{\rho_n}\{f\} = \sum_{i=1}^{n} W_{ni}f(\xi_{ni})$, where $\mathbb{X} = \{\xi_{ni}\} \subset \mathbb{T}$ is a node array.

Inspired by the results of the previous sections, our first guess is to choose the knots as the zeros of the para-orthogonal functions for the positive weight ω and construct interpolatory quadrature formulas in a subspace $\mathcal{R}_{p,q}$ of dimension n. For this kind of quadrature formulas, we can show that the coefficients do not grow too fast: There is an absolute constant C_3 such that $\sum_{j=1}^{n}|W_{nj}| \leq C_3\sqrt{n}$. Then, using a rational generalization of the Jackson III theorem (see [12, Theorem 4.7]), it can be shown that the following convergence result holds.

Theorem 6.1 *[12] Let ω and ρ be as in (6.1). Let $f \in C(\mathbb{T})$ with modulus of continuity $\omega(f,\delta) = O(\delta^p)$, $p > 1/2$. Let $p(n)$ and $q(n)$ be nonnegative integers with $p(n)+q(n) = n-1$ and $\lim_{n\to\infty} p(n)/n = 1/2$. Then the interpolatory quadrature formulas $I_{\rho_n}\{f\}$ whose knots are the zeros of the para-orthogonal function for ω and which are exact in $\mathcal{R}_{p(n),q(n)}$ converge to $I_\rho\{f\}$.*

Note that we need $p > 1/2$ so that convergence in $C(\mathbb{T})$ is not proved.

To get convergence for a larger class, we consider N-point quadrature formulas of interpolatory type in $\mathcal{R}_{n,n}$ with $N = 2n+1$. The basic idea for constructing such quadrature formulas is the following. Define $g(e^{i\theta}) = \rho(\theta)/\omega(\theta)$; then it is clear that

$$I_\rho\{f\} = \int_{-\pi}^{\pi} f(e^{i\theta})\rho(\theta)\mathrm{d}\theta = \int_{-\pi}^{\pi} f(e^{i\theta})g(e^{i\theta})\omega(\theta)\mathrm{d}\theta = I_\omega\{fg\}.$$

Now ω is positive and we can apply our previous theory of R-Szegő formulas. However, the integrand is now a product fg. If we want equality $I_\rho\{f\} = I_N^\rho\{f\}$, for $f \in \mathcal{R}_{n,n}$, then we must be able to integrate fg exactly for $f \in \mathcal{R}_{n,n}$. It can be shown that $I_\omega\{fg\} = I_\omega\{fg_{2n}\}$ for all $f \in \mathcal{R}_{n,n}$ if g_{2n} is the orthogonal projection of g onto $\mathcal{R}_{n,n}$ in L^2_ω [17]. Thus, it is sufficient to construct quadrature formulas exact in $\mathcal{R}_{n,n} \cdot \mathcal{R}_{n,n}$ so that $I_\omega\{fg\}$ and hence also $I_\rho\{f\}$ can be computed exactly for all $f \in \mathcal{R}_{n,n}$. Note that by the product $\mathcal{R}_{n,n} \cdot \mathcal{R}_{n,n}$, we double each pole.

Therefore, we associate with the sequence $\mathbb{A} = \{\alpha_k\}$ the doubling sequence $\tilde{\mathbb{A}} = \{0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \ldots\}$, denoted as $\{\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \ldots\}$. This doubling sequence can be used in exactly the same way as before to define Blaschke products $\tilde{B}_n$ and spaces $\tilde{\mathcal{L}}_n$, and orthogonal rational functions $\tilde{\varphi}_n$. The para-orthogonal rational functions $\tilde{Q}_n(z;\tau_n) = \tilde{\varphi}_n + \tau_n\tilde{\varphi}_n^*$ with $\tau_n \in \mathbb{T}$ and $\tilde{\varphi}_n^* = \tilde{B}_n\tilde{\varphi}_{n*}$ have n simple zeros ξ_{ni}, $i = 1, \ldots, n$, that can be used to construct R-Szegő quadrature formulas. Now set $N = 2n+1$ and let $\tilde{I}_\omega^N\{f\}$ be the R-Szegő formula that is exact in $\tilde{R}_{N-1,N-1} = \tilde{\mathcal{L}}_{N-1} \cdot \tilde{\mathcal{L}}_{(N-1)*}$. Since $F \in \tilde{\mathcal{R}}_{N-1,N-1} \Leftrightarrow F = fg$ with $f, g \in \mathcal{R}_{n,n}$ we have reached our objective. We have

Theorem 6.2 *[17] As in (6.1), let ρ be complex, ω positive, and $g(e^{i\theta}) = \rho(\theta)/\omega(\theta) \in L^2_\omega(\mathbb{T})$. For $N = 2n+1$, let $\{\xi_{Nj}\}_{j=1}^N$ be the zeros of the para-orthogonal function from $\tilde{\mathcal{L}}_N$ associated with the doubling sequence $\tilde{\mathbb{A}}$ and the weight ω. Moreover, let $\tilde{A}_{Nj}$ be the weights of the corresponding N-point R-Szegő formula, exact in $\tilde{\mathcal{R}}_{N-1,N-1}$. Then the quadrature formula $I_\rho^N\{f\} = \sum_{i=1}^N W_{Nj}f(\xi_{Nj})$ computes the integral $I_\rho\{f\}$ exactly for all $f \in \mathcal{R}_{n,n}$ if the weights are given by $W_{Nj} = \tilde{A}_{Nj}g_{2n}(\xi_{Nj})$, where g_{2n} is the projection of g onto $\mathcal{R}_{n,n}$ in $L^2_\omega(\mathbb{T})$.*

This defines the quadrature formulas $I_{\rho_n}\{f\}$. Now to prove convergence, we use a rational extension of the classical Erdős-Turán theorem: if $f_N \in \mathcal{R}_{n,n}$ interpolates f in the points $\{\xi_{Nk}\}_{k=1}^N$, then, under the conditions given in Theorem 6.2, f_N converges to f in $L^2_\omega(\mathbb{T})$. Using the bound $\sum_{k=1}^N |W_{Nk}| < C_3\sqrt{n}$, the Cauchy-Schwarz inequality, and a rational generalization of [42, Thm. 1.5.4], we get the following convergence result.

Theorem 6.3 *[17] Assume the same conditions and the same interpolatory quadrature formulas as in Theorem 6.2. If, moreover, $\sum_{j=1}^\infty (1 - |\alpha_j|) = \infty$, then the following convergence results hold.*
For any bounded f for which $I_\rho\{f\} < \infty$ exists as a Riemann integral, $I_\rho^N\{f\}$ converges to $I_\rho\{f\}$.
For any bounded Riemann integrable f, $\sum_{j=1}^N |W_{Nj}|f(\xi_{Nj})$ converges to $\int_{-\pi}^{\pi} f(e^{i\theta})|\rho(\theta)|\mathrm{d}\theta$.
The MRA $F_N(z) = I_\rho^N\{D(\cdot, z)\}$ interpolates the Riesz-Herglotz transform $F_\rho(z) = I_\rho\{D(\cdot, z)\}$ at the points $\{0, \alpha_1, \ldots, \alpha_n\}$ and $\{\infty, 1/\overline{\alpha}_1, \ldots, 1/\overline{\alpha}_n\}$, and it converges to F_ρ uniformly on compact subsets on $\hat{\mathbb{C}} \setminus \mathbb{T}$.

The previous results are related to interpolatory quadrature formulas in $\mathcal{R}_{n,n}$. We can, however, generalize to the asymmetric case and consider more general spaces $\mathcal{R}_n = \mathcal{R}_{p(n),q(n)}$, where $p(n)$ and $q(n)$ are nondecreasing sequences of nonnegative integers such that $p(n)+q(n)=n-1$ and $\lim_{n\to\infty} p(n)/n = r \in (0,1)$. Note that the spaces $\mathcal{R}_n$ have dimension n and they are nested. As before, we need to introduce an asymmetric doubling sequence as follows. Set $r(n)=\max\{p(n),q(n)\}$, $s(n)=\min\{p(n),q(n)\}$, $\alpha_0=0$, $\tilde{\mathbb{A}}_n = \{\alpha_0,\alpha_1,\alpha_1,\ldots,\alpha_{s(n)},\alpha_{s(n)},\alpha_{s(n)+1},\ldots,\alpha_{r(n)}\} = \{\tilde\alpha_0,\tilde\alpha_1,\ldots,\tilde\alpha_{n-1}\}$. Since increasing n to $n+1$ increases either $p(n)$ or $q(n)$ by one, this increases either $r(n)$ or $s(n)$ by one. The numbering of the $\tilde\alpha_k$ is such that $\tilde\alpha_n$ is either a repeated point $\alpha_{s(n)+1}$ or a new point $\alpha_{r(n)+1}$. This defines the sequence $\tilde{\mathbb{A}} = \{\tilde\alpha_1,\tilde\alpha_1,\ldots\}$ uniquely. The quantities related to the $\tilde{\mathbb{A}}$ are as before denoted with a tilde. We construct quadrature formulas whose nodes are the zeros ξ_{ni} of the para-orthogonal function $\tilde{Q}_n(z;\tau_n)=\tilde\varphi_n(z)+\tau_n\tilde\varphi_n^*(z)$. The $\tilde\varphi_n \in \tilde{\mathcal{L}}_n\setminus\tilde{\mathcal{L}}_{n-1}$ are the orthogonal functions with respect to the positive measure ω with the properties introduced before. The weights W_{nk} of these quadrature formulas $I_{\rho_n}\{f\}=\sum_{k=1}^n W_{nk}f(\xi_{nk})$ are constructed such that $I_{\rho_n}\{f\}$ is exact in $\mathcal{R}_n=\mathcal{R}_{p(n),q(n)}$ of dimension n. With this setting, one can follow the same approach as in Theorem 5.4, but now the MA's are replaced by MRA's of order $(p(n)+1,q(n)+1)$.

Theorem 6.4 *[18] Assume $\int_{-\pi}^{\pi}\log\omega(\theta)\mathrm{d}\theta > -\infty$ and let the sequence $\mathbb{A}$, hence also $\tilde{\mathbb{A}}$, be included in a compact subset of $\mathbb{D}$. Denote by F_{ρ_n} the MRA of order $(p(n)+1,q(n)+1)$ to the Riesz-Herglotz transform F_ρ. The denominator of F_{ρ_n} is $\prod_{k=1}^n(z-\xi_{nk})$, where the ξ_{nk} are the zeros of the para-orthogonal functions in $\tilde{\mathcal{L}}_n$ with respect to the sequence $\tilde{\mathbb{A}}$ and the positive function ω. This sequence $\tilde{\mathbb{A}}$ is defined as above in terms of the sequence $\mathbb{A}$ and the sequences of integers $(p(n),q(n))$, $p(n)+1(n)=n-1$. The functions ρ and ω satisfy (6.1). Then the following convergence results hold:*
$\limsup_{n\to\infty}|F_\rho(z)-F_{\rho_n}(z)|^{1/n} \le \exp\{r\lambda(z)\} < 1$, $\forall z\in\mathbb{D}$,
$\limsup_{n\to\infty}|F_\rho(z)-F_{\rho_n}(z)|^{1/n} \le \exp\{s\lambda(\hat z)\} < 1$, $\forall z\in\mathbb{E}$, where $\hat z = 1/\overline{z}$,
where $r=\lim_{n\to\infty}p(n)/n$, $s=\lim_{n\to\infty}q(n)/n = 1-r$, and $\lambda(z)$ as in (5.1).

From this theorem, the following theorem follows directly by using Theorem 2.2.

Theorem 6.5 *[18] Under the same conditions as in the previous theorem, assume the quadrature formulas $I_{\rho_n}\{f\}$ have nodes $\{\xi_{nk}\}_{k=1}^n$ and their weights are defined such that the formulas are exact in $\mathcal{R}_{p(n),q(n)}$ of dimension n. Then it holds that*

$$\limsup_{n\to\infty}|I_\rho\{f\}-I_{\rho_n}\{f\}|^{1/n} \le \gamma < 1$$

for any function f analytic in a closed region $\mathbb{G}$ such that $\mathbb{T}\subset\mathbb{G}\subset\mathbb{C}\setminus(\mathbb{A}\cup\hat{\mathbb{A}})$, where $\gamma=\max\{\gamma_1,\gamma_2\}$ with

$$\gamma_1 = \max_{z\in\Gamma\cap\mathbb{D}}\exp\{r\lambda(z)\} \quad \textit{and} \quad \gamma_2 = \max_{z\in\Gamma\cap\mathbb{E}}\exp\{s\lambda(\hat z)\},$$

$\lambda(z)$ as in (5.1), $\Gamma=\partial\mathbb{G}$ is the boundary of $\mathbb{G}$ consisting of finitely many rectifiable curves, $r=\lim_{n\to\infty}p(n)/n$, $s=\lim_{n\to\infty}q(n)/n=1-r$, $\hat z = 1/\overline{z}$, $\hat{\mathbb{A}}=\{\hat\alpha = 1/\overline{\alpha} : \alpha\in\mathbb{A}\}$.

Note that if we take the balanced situation, i.e., when $r=s=1/2$, then from this theorem it follows that $\gamma=\sqrt{\tilde\gamma}$ with $\tilde\gamma=\max\{\tilde\gamma_1,\tilde\gamma_2\}$, where $\tilde\gamma_1=\max_{z\in\Gamma\cap\mathbb{D}}\exp\{\lambda(z)\}$ and $\tilde\gamma_2 = \max_{z\in\Gamma\cap\mathbb{E}}\exp\{\lambda(\hat z)\}$. If we assume that ρ is positive, then we can take $\omega=\rho$ and $\tilde{\mathbb{A}}=\mathbb{A}$, so that the quadrature formulas then considered in this theorem are precisely the R-Szegő formulas. This result is confirmed by Theorem 5.6, where indeed the bound $\tilde\gamma$ is given. The squaring $\gamma=\tilde\gamma^2$ is of course to be expected.

7 Poles in the support of the measure

So far, we have assumed that the poles of the rational functions were outside the support of the measure. If the poles are selected in the support, then we can refer to the theory of orthogonal rational functions with poles on $\mathbb{T}$ when we want to compute integrals over $\mathbb{T}$. This theory is analogous and yet different from what was explained in Sections 3-6. It generalizes the differences that also exist between polynomials orthogonal on the real line and polynomials orthogonal on the unit circle.

So instead of choosing points α_k inside $\mathbb{D}$, we choose them all on the boundary $\mathbb{T}$. We need to define one exceptional point on $\mathbb{T}$ that is different from all α_k. We assume without loss of generality that it is -1. So $\mathbb{A} = \{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{T} \setminus \{-1\}$. We consider the spaces $\mathcal{L}_n = \operatorname{span}\{1/\omega_0, 1/\omega_1, \ldots, 1/\omega_n\}$, where $\omega_k(z) = \prod_{i=1}^{k}(z - \alpha_i)$ as before. The theory can be developed along the same lines, but it is a bit more involved. We use the same notation where possible. Now it is important that if $\varphi_n(z) = p_n(z)/\pi_n(z)$, then $p_n(\alpha_{n-1}) \neq 0$. If this holds, then φ_n is called regular, and the system $\{\varphi_n\}$ is regular if every function in the system is regular. It is for such a regular system that one can prove that the orthogonal functions satisfy a recurrence relation of the form [10]

$$\varphi_n(z) = \frac{A_n}{z - \alpha_n}\varphi_{n-1}(z) + B_n\frac{z - \alpha_{n-2}}{z - \alpha_n}\varphi_{n-1}(z) + C_n\frac{z - \alpha_{n-2}}{z - \alpha_n}\varphi_{n-2}(z), \quad n = 2, 3, \ldots, \quad \alpha_0 = 0.$$

These constants satisfy $A_n + B_n(\alpha_{n-1} - \alpha_{n-2}) \neq 0$ and $C_n \neq 0$ for $n = 2, 3, \ldots$.

The para-orthogonal functions are in this case replaced by quasi-orthogonal functions. These are defined as

$$Q_n(z; \tau_n) = \varphi_n(z) + \tau_n \frac{(1 + \alpha_n)(z - \alpha_{n-1})}{(1 + \alpha_{n-1})(z - \alpha_n)}\varphi_{n-1}(z), \quad \tau_n \in \mathbb{R}.$$

We have

Theorem 7.1 *[10] If the system $\{\varphi_n\}$ is regular, then it is always possible to find (infinitely many so-called regular values) $\tau_n \in \mathbb{R}$ such that the quasi-orthogonal functions $Q_n(z; \tau_n)$ have precisely n zeros, all simple and on $\mathbb{T} \setminus \{\alpha_1, \ldots, \alpha_n\}$.*

Let $\{\xi_{nk}\}_{k=1}^{n}$ be these zeros. If we take them as knots for an interpolating quadrature formula for $\mathcal{L}_{n-1}$, then this quadrature formula will have positive weights and it will be exact in $\mathcal{L}_{n-1} \cdot \mathcal{L}_{n-1}$. If $\tau_n = 0$ is a regular value, then the corresponding quadrature formula is exact in $\mathcal{L}_{n-1} \cdot \mathcal{L}_n$.

These quadrature formulas are the analogs of the R-Szegő formulas. We shall denote them again by $I_{\mu_n}\{f\} = \sum_{k=1}^{n} A_{nk} f(\xi_{nk})$. In fact, one can, exactly as for the R-Szegő formulas, express A_{nk} and ξ_{nk} in terms of the reproducing kernels. Note that A_{nk} and ξ_{nk} depend as before on the choice of τ_n. Then it can be shown that $I_{\mu_n}\{D(\cdot, z)\} = F_{\mu_n}(z) = -P_n(z; \tau_n)/Q_n(z; \tau_n)$. This is a rational approximant for the Riesz-Herglotz transform $F_\mu(z) = I_\mu\{D(\cdot, z)\}$ in a particular sense. Indeed, the $F_{\mu_n}(z)$ and $F_\mu(z)$ are defined for $z \in \mathbb{D}$, but extending them by a nontangential limit to the boundary $\mathbb{T}$, then we have interpolation in $\{0, \infty, \alpha_1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1}\}$ (repeated points imply interpolation in the Hermite sense). In case $\tau_n = 0$ is a regular value, then $F_{\mu_n} = \psi_n/\varphi_n$, with ψ_n as before the functions of the second kind associated with φ_n. Then this F_{μ_n} will also interpolate in the extra point α_n. It can be shown that if $\{\varphi_n\}$ is a regular system, then there is a subsequence $F_{\mu_{n(s)}}$ that converges to F_μ locally uniformly in $\mathbb{C} \setminus \mathbb{T}$. However convergence has been explored only partially, and here is a wide open domain for future research.

8 Integrals over an interval

By conformally mapping the unit circle to the real line, we can obtain analogous results on the real line. The results have different formulations, but they are essentially the same as the ones we gave in the previous sections. We consider instead some other quadrature formulas that were derived, making use of rational functions. We restrict ourselves in the first place to a compact interval Δ on the real line, which we can always renormalize to be $[-1, 1]$.

So we now consider measures that are supported on an interval of the real line and we assume that this interval is $\operatorname{supp}(\mu) \subset \Delta = [-1, 1]$. The problem is to approximate the integral $\int_\Delta f(x)dx$. Several quadrature formulas of the form $\sum_{k=1}^{n} A_k f(x_k)$, exact for other functions than polynomials have been considered in the literature before. We shall discuss formulas exact for spaces of rational functions with prescribed poles outside Δ. For more general cases, see the classical book [22, p. 122] and references therein.

Consider a positive measure μ. In [25], Gautschi considers the following problem. Let α_k, $k = 1, \ldots, M$ be distinct numbers in $\mathbb{C} \setminus \Delta$. For given integers m and n, with $1 \leq m \leq 2n$, find an n-point quadrature formula exact for all monomials x^j, $j = 0, \ldots, 2n - m - 1$, as well as for the rational functions $(x - \alpha_k)^{-s}$, $k = 1, \ldots, M$, $s = 1, \ldots, s_k$, with $s_k \geq 1$ and $\sum_{k=1}^{M} s_k = m$. The solution is given in the following theorem (which is also valid for an unbounded support Δ).

Theorem 8.1 *[25] Given a positive measure μ with* $\operatorname{supp}(\mu) \subset \Delta \subset \mathbb{R}$, $\{\alpha_k\}_{k=1}^{M} \subset \mathbb{C} \setminus \Delta$, *and positive integers s_k, $\sum_{k=1}^{M} s_k = m$, define $\omega_m(x) = \prod_{k=1}^{M}(x - \alpha_k)^{s_k} \in \Pi_m$. Assume that the measure $d\mu/\omega_m$ admits a (polynomial) n-point Gaussian quadrature formula, i.e., there are $\xi_j^G \in \Delta$ and $A_j^G > 0$ such that*

$$\int_\Delta f(x)\frac{d\mu(x)}{\omega_m(x)} = \sum_{j=1}^{n} A_j^G f(\xi_j^G) + E_n^G\{f\} \quad \textit{with} \quad E_n^G\{f\} = 0, \quad \forall f \in \Pi_{2n-1}.$$

Define $\xi_j = \xi_j^G$ and $A_j = A_j^G \omega_m(\xi_j^G)$, $j = 1, \ldots, n$. Then $\int_\Delta f(x)d\mu(x) = \sum_{j=1}^{M} A_j f(\xi_j) + E_n\{f\}$, where $E_n\{f\} = 0$ for all $f \in \Pi_{2n-m-1}$ and for all $f \in \{(x-\alpha_k)^{-s} : k = 1, \ldots, M; s = 1, \ldots, s_k\}$.

Depending on the application, several special choices of $\{\alpha_k\}$ are proposed: they may contain real numbers and/or complex conjugate pairs, and they may be of order 1 or of order 2. Independently, Van Assche and Vanherwegen [43] discuss two special cases of Theorem 8.1: the α_k are real and either all $s_k = 1$ and $m = 2n$ (a polynomial of degree -1 is understood as identically zero), or all but one $s_k = 2$ and $m = 2n - 1$. The first case is called "Gaussian quadrature", the second "orthogonal quadrature".

The main observation to be made with these quadrature formulas is that the nodes and weights are closely related to the zeros and Christoffel numbers for polynomials orthogonal on Δ with respect to a varying measure. This interaction, also observed by López and Illan [33, 34], makes it possible to use results from orthogonal polynomials to get useful properties for the nodes and weights for quadrature based on rational interpolation. This is the main contribution of [43] along with the convergence in the class of continuous functions. It should be pointed out that unlike [25, 43], in [33, 34] non-Newtonian tables of prescribed poles are used, so that when considering convergence results, some additional conditions on the poles are necessary in order to assure the density of the rational functions that are considered in the space $C(\Delta)$ of functions continuous in Δ. For instance, when all the α_k are different

$$\sum_{k=1}^{\infty}(1-|c_k|)=\infty \quad \text{where} \quad c_k=\alpha_k-\sqrt{\alpha_k^2-1} \tag{8.1}$$

(see [1, p. 254]).

We also mention here the work of Gautschi [26, 27, 28]. There, Gauss-type formulas are derived that are exact on a set of rational functions whose poles are prescribed but where the numerator degree can be larger than the denominator degree. The idea is again to consider the denominator as part of the measure and construct the usual Gauss quadrature formulas for the modified measure. In the case of the Lebesgue measure, such formulas are also considered in [44].

Finally, we should mention the works of G. Min [38, 39], where also quadrature formulas based on rational functions are considered when taking $d\mu(x)=dx/\sqrt{1-x^2}$, $x\in\Delta$. The author makes use of the properties of the generalized Chebyshev "polynomials" associated with the rational system

$$\left\{1,\frac{1}{x-\alpha_1},\frac{1}{x-\alpha_2},\ldots,\frac{1}{x-\alpha_n}\right\}, \quad n=1,2,\ldots, \quad x\in\Delta. \tag{8.2}$$

This generalized notion is introduced in [2]. The term *polynomial* is misleading because they are in fact *rational* functions in the span of the functions (8.2). The qualification *Chebyshev* is justified by the fact that they have properties similar to classical Chebyshev polynomials. Using these Chebyshev functions and assuming that $\{\alpha_k\}_{k=1}^n\subset\mathbb{R}\setminus\Delta$, $n=1,2,\ldots$, G. Min constructs the n-point interpolatory quadrature formula

$$Q_n\{f\}=\sum_{k=1}^{n}A_kf(\xi_k)\approx\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}dx,$$

in the zeros of the generalized Chebyshev "polynomials", and it turns out that this formula is exact for all functions $f\in\operatorname{span}\{(x-\alpha_1)^{-1},(x-\alpha_1)^{-2},\ldots,(x-\alpha_n)^{-1},(x-\alpha_n)^{-2}\}=\mathcal{R}_{2n-1}(\alpha_1,\ldots,\alpha_n)$.

Theorem 8.2 *[38] Let the $\{\alpha_k\}$ and $Q_n\{f\}$ be defined as above. Let $\{\xi_k\}_{k=1}^n$ be the zeros of $T_n(x)$, the generalized Chebyshev "polynomial" associated with (8.2). Then (a) $Q_n\{f\}$ is a positive quadrature formula (i.e. $A_k>0$ for $k=1,\ldots,n$) and (b) $\int_{-1}^{1}f(x)/\sqrt{1-x^2}dx=Q_n\{f\}$ for any $f\in\mathcal{R}_{2n-1}(\alpha_1,\ldots,\alpha_n)$.*

Since the converse of Theorem 8.1 is also true, it follows that the zeros of the nth Chebyshev polynomial for (8.2) coincide with the zeros of the orthonormal polynomial of degree n with respect to the varying measure

$$d\mu(x)=\frac{1}{\sqrt{1-x^2}(x-\alpha_1)^2\cdots(x-\alpha_n)^2}, \quad \{\alpha_j\}_{j=1}^n\subset\mathbb{R}\setminus\Delta.$$

On the other hand, let U_n be the Chebyshev polynomial of the second kind associated with (8.2). It is known [3, Thm. 1.2] that (a) $T_n^2(x)+\left(\sqrt{1-x^2}U_n(x)\right)^2=1$, (b) there are $n+1$ points $\{\tilde{\xi}_k\}$ with $-1=\tilde{\xi}_n<\tilde{\xi}_{n-1}<\cdots<\tilde{\xi}_1<\tilde{\xi}_0=1$ such that $T(\tilde{\xi}_k)=(-1)^k$, $k=0,\ldots,n$. Since $\|T_n\|_{[-1,1]}=1$, $\{\tilde{\xi}_k\}_0^n$ are the extreme points of T_n and also $\{\tilde{\xi}_k\}_1^{n-1}$ are the zeros of U_n.

Theorem 8.3 *[38] Let the elements* $\{\alpha_k\}_1^n \subset \mathbb{C}\setminus\mathbb{R}$ *be paired by complex conjugation and let* $\{\tilde{\xi}_k\}_1^{n-1}$ *be the zeros of* $U_n(x)$ *as defined above. Then there exist positive parameters* $\tilde{A}_0,\ldots,\tilde{A}_n$ *such that*

$$\tilde{Q}_n\{f\} = \tilde{A}_0 f(1) + \sum_{k=1}^{n-1} \tilde{A}_k f(\tilde{\xi}_k) + \tilde{A}_n f(-1) = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx, \quad \forall f \in \mathcal{R}_{2n-1}(\alpha_1,\ldots,\alpha_n).$$

This is a Lobatto-type quadrature formula.

Let us next assume that μ is a complex measure in $\Delta = [-1,1]$. The Theorem 8.1 is still valid, however some difficulties have to be addressed. We need to guarantee the existence of n-point Gaussian quadrature formulas for a measure of the type $d\mu(x)/\omega_m(x)$ as defined in Theorem 8.1. This requires orthogonal polynomials with respect to a complex measure, and these need not be of degree n, and if they are, their zeros can be anywhere in the complex plane. In [30] the authors could rely on known results about the asymptotic behavior of polynomials orthogonal with respect to fixed complex measures and their zeros to overcome these difficulties. For a general rational setting, a treatment similar to the one in Section 6 is given in [29, 21, 20]. The idea is as follows. Assume $d\mu(x) = \rho(x)dx$ with $\rho(x) \in L^1(\Delta)$, possibly complex. Let $\mathbb{A}_n = \{\alpha_{jn} : j = 1,\ldots,n\}$ and $\mathbb{A} = \bigcup_{n\in\mathbb{N}} \mathbb{A}_n$ with $\mathbb{A} \subset \hat{\mathbb{C}}\setminus\Delta$ be given, and set $\omega_n(x) = (x-\alpha_{1n})\cdots(x-\alpha_{nn})$. For each n, define the space $\mathcal{R}_n = \{P(x)/\omega_n(x) : P \in \Pi_{n-1}\}$. Given n distinct points $\{\xi_{1n},\ldots,\xi_{nn}\} \subset \Delta$, there exist parameters $A_{1n},\ldots,A_{nn}$ such that

$$I_\rho\{f\} := \int_{-1}^{1} f(x)\rho(x)dx = I_{\rho_n}\{f\} := \sum_{j=1}^{n} A_{jn} f(\xi_{jn}), \quad \forall f \in \mathcal{R}_n.$$

We call $I_{\rho_n}\{f\}$ an n-interpolatory quadrature formula for $\mathcal{R}_n$. By introducing an auxiliary positive weight function $\beta(x)$ on Δ and taking $\{\xi_{jn}\}_{j=1}^n$ as the zeros of the nth orthogonal polynomial with respect to $\beta(x)/|\omega_n(x)|^2$, several results on the convergence for these quadrature formulas have been proved. González-Vera et al. prove in [29] the convergence of this type of quadrature formulas in the class of continuous functions satisfying a certain Lipschitz condition. Cala-Rodriguez and López-Lagomasino in [21] derive exact rates of convergence when approximating Markov-type analytic functions. In both of these papers, the intimate connection between multipoint Padé-type approximants and interpolatory quadrature formulas is explicitly exploited. The same kind of problem is considered in [20] from a purely "numerical integration" point of view. The most relevant result is

Theorem 8.4 *[20] Set* $\Delta = [-1,1]$, $\mathbb{A} = \bigcup_{n\in\mathbb{N}} \mathbb{A}_n \subset \hat{\mathbb{C}}\setminus\Delta$ *with* $\mathbb{A}_n = \{\alpha_{jn} : j = 1,\ldots,n\}$. *Assume that* $\operatorname{dist}(\mathbb{A},\Delta) = \delta > 0$ *and that for each* $n \in \mathbb{N}$ *there exists an integer* $m = m(n)$, $1 \le m \le n$, *such that* $\alpha_m \in \mathbb{A}_n$ *satisfies* $|\mathrm{Re}\,(\alpha_m)| > 1$. *Let* $\rho(x) \in L^1(\Delta)$ *and* $\beta(x) \ge 0$ *on* Δ *such that* $\int_\Delta |\rho(x)|^2/\beta(x)dx = K_1^2 < \infty$. *Let* $I_{\omega_n}\{f\} = \sum_{j=1}^n A_{jn} f(\xi_{jn})$ *be the* n*-point interpolatory quadrature formula in* $\mathcal{R}_n$ *for the nodes* $\{\xi_{jn}\}$ *that are zeros of* $Q_n(x)$, *the* n*th orthogonal polynomial with respect to* $\beta(x)/|\omega_n(x)|^2$, $x \in \Delta$. *Then,* $\lim_{n\to\infty} I_{\rho_n}\{f\} = I_\rho\{f\}$ *for all bounded complex valued functions on* Δ *such that the integral* $I_\rho\{f\}$ *exists.*

As we have seen, when dealing with convergence of sequences of quadrature formulas based on rational functions with prescribed poles, one needs some kind of condition about the separation of the poles and the support of the measure (positive or complex) like for example in Theorem 8.4 or some other condition like (8.1). Thus, it is a natural question to ask what happens when sequences of points in the table $\mathbb{A}$ tend to supp $(\mu) \subset \Delta$ or when some points are just chosen in Δ. Consider for example the situation where the points in $\mathbb{A}$ are just a repetition of the boundary points -1 or $+1$ of the interval $\Delta = [-1, 1]$. According to the approach given in [37], let us consider the transformation $\varphi : [-1, 1] \to [0, \infty]$ given by $t = \varphi(x) = (1+x)/(1-x)$. Thus, after this change of variables, we can pass from an integral $\int_{-1}^{1} f(x) d\mu(x)$ to an integral $\int_{0}^{\infty} g(t) d\lambda(t)$. The poles at $x = 1$ are moved to poles at $t = \infty$ and the poles at $x = -1$ are moved to poles at $t = 0$. This means that our rational functions are reduced to Laurent polynomials. This special situation is closely related to the so-called strong Stieltjes moment problem (see Section 9.3). The L-polynomials appear in two-point Padé approximants in a situation similar to what was discussed in Section 2. However, the difference is that now 0 and ∞ are points in the support of the measure. The generalization is that we consider a sequence of poles α_k that are in the support of the measure. Then we are back in the situation similar to the one discussed in Section 7.

9 Open problems

Several open problems have been explicitly mentioned or at least been hinted to in the text, and others may have jumped naturally to the mind of attentive readers. We add a few more in this section.

9.1 Error bounds

In this paper we have considered the convergence of modified approximants (MA) or multi-point rational approximants (MRA) and the corresponding convergence of R-Szegő quadrature formulas or interpolatory quadrature formulas. We have used some error bounds both for the approximants and the quadrature formulas (and these are closely related by Theorem 3.5). However, we did not give sharp bounds, and this is of course essential for computing exact rates of convergence.

Using continued fraction theory, Jones and Waadeland [36] recently gave computable sharp error bounds for MAs, in the polynomial case, i.e., $\alpha_k = 0$ for all k. A similar treatment could be done in the rational case. As an illustration, we consider the case of the normalized Lebesgue measure $d\mu(\theta) = \mathrm{d}\theta/(2\pi)$. Let $R_{\mu_n}(z;\tau_n) = F_\mu(z) - F_{\mu_n}(z)$ be the error for the MA. Recall (3.3), where $p + q = n - 1$, $X_n(z) = \prod_{k=1}^{n}(z - \xi_{nk})$, with ξ_{nk} the zeros of the para-orthogonal function $Q_n(z;\tau_n)$. Let us take $p = 0$ and $q = n - 1$; then for the normalized Lebesgue measure we have

$$R_{\mu_n}(z) = \frac{2z\pi_{n-1}(z)}{X_n(z)} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{X_n(t)\mathrm{d}t}{\pi_{n-1}(t)(t-z)t}.$$

If $z \in \mathbb{D}$, then by the residue theorem, one gets

$$R_{\mu_n}(z) = 2\left[1 - \frac{X_n(0)\pi_{n-1}(z)}{X_n(z)}\right].$$

Now using the choice $\tau_n = \eta_n$ as in Example 4.1, we have

$$Q_n(z;\tau_n) = \frac{X_n(z)}{\pi_n(z)}, \quad X_n(z) = c\prod_{k=1}^{n}(z - \xi_{nk}) = \kappa_n\tau_n[z\omega_{n-1}(z) + \pi_{n-1}(z)].$$

Therefore,

$$R_{\mu_n}(z) = 2\left[1 - \frac{\pi_{n-1}(z)}{z\omega_{n-1}(z) + \pi_{n-1}(z)}\right] = \frac{2z\omega_{n-1}(z)}{z\omega_{n-1}(z) + \pi_{n-1}(z)}, \quad z \in \mathbb{D}.$$

This is an example of an explicit expression for the error of approximation. It is an open problem to extend this to the general case.

9.2 Exact rates of convergence

In Theorem 5.5 we obtained estimates for the rate of convergence. It is however not clear what conditions on $\mathbb{A}$ and μ have to be imposed so as to obtain equality, i.e., when a formula of the form $\limsup_{n\to\infty} |R_{\mu_n}(z)|^{1/n} = \exp\{\lambda(z)\}$ holds.

If μ is the normalized Lebesgue measure and all $\alpha_k = 0$, then $R_{\mu_n}(z) = 2z^n/(z^n+1)$, and therefore one gets

$$\lim_{n\to\infty} |R_{\mu_n}(z)|^{1/n} = |z| = \exp\{\lambda(z)\}, \quad z \in \mathbb{D}.$$

Is it true for the normalized Lebesgue measure and $\mathbb{A}$ included in a compact subset of $\mathbb{D}$ that

$$\lim_{n\to\infty} \left|\frac{2z\omega_{n-1}(z)}{z\omega_{n-1}(z) + \pi_{n-1}(z)}\right|^{1/n} = \exp\{\lambda(z)\},$$

where $\lambda(z)$ is as in (5.1)?

9.3 Stieltjes problems

Concerning Stieltjes and strong Stieltjes moment problems, several situations are considered that correspond to special choices of the poles like a finite number of poles that are cyclically repeated. Several results were obtained concerning the moment problem and the multipoint Padé approximants. See for example [16] for the rational moment problem where poles are allowed on the unit circle. The existence proof given there is closely related to the convergence of the quadrature formulas. Several other papers exist about convergence of multipoint Padé approximants with or without a cyclic repetition of the poles. The quadrature part is still largely unexplored. In [37], this situation is briefly mentioned.

For the case of L-polynomials where one considers only the poles 0 and ∞, the convergence of two-point Padé approximants to Stieltjes transforms was studied in [31] and in [5, 6]. The latter papers also give error estimates and consider the corresponding convergence and rates of convergence of the quadrature formulas. To illustrate this, we formulate some of the theorems.

Theorem 9.1 *[6] Let μ be a positive measure on $\Delta = [a,b]$ with $0 \le a < b \le \infty$. Let $Q_n^\mu(x) = \kappa_n \prod_{k=1}^n (x - \xi_{jn})$, $\kappa_n > 0$, be the nth orthonormal polynomial with respect to $x^{-p}d\mu(x)$, $p \ge 0$. Let $F_\mu(z) = I_\mu\{(x-z)^{-1}\} = \int_\Delta \frac{1}{x-z} d\mu(x)$ be the Cauchy transform of μ. Let $I_{\mu_n}\{f\} = \sum_{j=1}^n A_{jn} f(\xi_{jn})$ be the n-point Gaussian formula in $\Lambda_{-p,q}$ where $0 \le p \le 2n$, $q \ge -1$ and $p+q = 2n-1$, and set $F_{\mu_n}(z) = I_{\mu_n}\{(x-z)^{-1}\}$. Then F_{μ_n} is a rational function of type $(n-1,n)$ that is a two-point Padé approximant (2PA) for F_μ (order p at the origin and order $q+2$ at infinity).*

López-Lagomasino et al. prove in several papers (see for example [37]) the uniform convergence of the 2PA in compact subsets of $\mathbb{C}\setminus\Delta$ and give estimates for the rate of convergence. They assume some Carleman type conditions, namely either $\lim_{n\to\infty} p(n)=\infty$ and $\sum_{n=1}^{\infty} c_{-n}^{-1/2n}=\infty$ or $\lim_{n\to\infty}[2n-1-p(n)]=\infty$ and $\sum_{n=1}^{\infty} c_n^{-1/2n}=\infty$, where the moments are defined as $c_n=\int x^n d\mu(x)$, $n\in\mathbb{Z}$.

When the measure $d\mu(x)=\rho(x)dx$ is complex, with $\int_\Delta |\rho(x)|dx<\infty$, then an auxiliary positive measure $\omega(x)dx$ with $\omega(x)>0$, $x\in\Delta$, is introduced such that $\int_\Delta |\rho(x)|^2/\omega(x)dx = K^2<\infty$.

Theorem 9.2 *[6] Let Q_n^ω be the nth orthogonal polynomial with respect to $x^{-2p}\omega(x)$ whose zeros are $\xi_{jn}\in\Delta$, and let $I_{\mu_n}\{f\}=\sum_{j=1}^n A_{jn}f(\xi_{jn})$ be the interpolatory quadrature formula exact in $\Lambda_{-p,q}$, $p+q=n-1$. Then $F_{\mu_n}(z)=I_{\mu_n}\{(x-z)^{-1}\}$ is a rational function of type $(n-1,n)$ and it is a two-point Padé-type (2PTA) approximant for F_μ.*

Let $d_k=\int_\Delta x^k\omega(x)dx$, $k\in\mathbb{N}$, be the moments of ω and assume that $p=p(n)$ and $q=q(n)=n-1-p(n)$, such that either $\lim_{n\to\infty} p(n)=\infty$ and $\sum_{n=1}^{\infty} d_{-n}^{-1/2n}=\infty$ or $\lim_{n\to\infty} q(n)=\infty$ and $\sum_{n=1}^{\infty} d_n^{-1/2n}=\infty$. Then the 2PTA $F_{\mu_n}(z)$ converge to $F_\mu(z)$ uniformly in compact subsets of $\mathbb{C}\setminus\Delta$. The quadrature formula converges to the integral for all $f\in C^B[0,\infty)=\{f\in C[0,\infty):\lim_{x\to\infty} f(x)=L\in\mathbb{C}\}$ if and only if $\sum_{k=1}^n |A_{kn}|\le M$ for $n\in\mathbb{N}$.

Note that if in this theorem $d\mu$ is a positive Borel measure, we can set $\omega=\rho$, so that the quadrature formula becomes the n-point Gaussian formula, and then the Carleman conditions on its moments imply the converergence of the quadrature formulas in the class $C^B[0,\infty)$.

As for the rate of convergence, we assume that $\lim_{n\to\infty} p(n)/n=r\in[0,1]$, and we assume that μ is of the form $d\mu(x)=x^\nu\exp(-\tau(x))dx$, $\nu\in\mathbb{R}$, $\tau(x)$ continuous on $(0,\infty)$ and for $\gamma>1/2$ and $s>0$: $\lim_{x\to0^+}(sx)^\gamma\tau(x)=\lim_{x\to\infty}(sx)^{-\gamma}\tau(x)=1$. Set $D(\gamma)=\frac{2\gamma}{2\gamma-1}[\frac{\Gamma(\gamma+1/2)}{\sqrt{\pi}\Gamma(\gamma)}]^{1/2\gamma}$, where Γ is the Euler Gamma function, and with $\theta=1-1/(2\gamma)<1$, $\delta(z)=(1-r)^\theta\mathrm{Im}\,((sz)^{1/2})+r^\theta\mathrm{Im}\,((sz)^{-1/2})$, where the branch is taken such that $(-1)^{1/2}=i$. Furthermore, for f analytic in the domain $\mathbb{G}$ such that $[0,\infty)\subset\mathbb{G}\subset\hat{\mathbb{C}}$ and some compact K, define $\lambda(K)=\exp(-R)$ with $R=2D(\gamma)\inf_{z\in K}\{\delta(z)\}>0$ for some compact K. It is then proved, using results from [37], that

Theorem 9.3 *[6] With the notation just introduced, let $I_{\mu_n}\{f\}$ be the n-point Gaussian quadrature formula exact in $\Lambda_{-p(n),2n-1-p(n)}$ with error $E_{\mu_n}=I_\mu-I_{\mu_n}$. Let F_{μ_n} be the corresponding 2PA for the Cauchy transform F_μ and $R_{\mu_n}=F_\mu-F_{\mu_n}$ the associated error. Then we have that $\lim_{n\to\infty}\|R_{\mu_n}\|_K^{1/(2n)^\theta}=\lambda(K)$, where K is a compact subset of $\mathbb{C}\setminus[0,\infty)$ and $\|\cdot\|_K$ is the supremum norm in K. Also $\lim_{n\to\infty}E_{\mu_n}\{f\}=0$ for all f analytic in the domain $\mathbb{G}$, and $\limsup_{n\to\infty}|E_{\mu_n}\{f\}|^{1/(2n)^\theta}\le\lambda(\mathbb{J})<1$, where $\mathbb{J}\subset\mathbb{G}$ is a Jordan curve.*

If I_{μ_n} is the interpolatory quadrature formula of Theorem 9.2 and F_{μ_n} the corresponding 2PTA, then $R_{\mu_n}\to0$ uniformly in compact subsets K of $\mathbb{C}\setminus[0,\infty)$ and $\lim_{n\to\infty}\|R_{\mu_n}\|_K^{1/(2n)^\theta}=\sqrt{\lambda(K)}<1$. For the quadrature formula it holds that $\limsup_{n\to\infty}|E_{\mu_n}\{f\}|^{1/n^\theta}\le\sqrt{\lambda(\mathbb{J})}<1$ with $\mathbb{J}\subset\mathbb{G}$ a Jordan curve and f analytic in $\mathbb{G}$.

It is still an open problem to generalize this kind of results to the multipoint case where we select a number of poles $\alpha_k\in[0,\infty]$. Also the multipoint problem corresponding to Hamburger moment problem (the measure is supported on the whole of $\mathbb{R}$ as explained in section 7) needs generalization. There is almost nothing published about error estimates, convergence or rate of convergence for the rational approximants or for the quadrature formulas.

9.4 Miscellaneous problems

1. In the convergence results where the poles of the rational functions are outside the support of the measure, it was assumed that they stayed away (they were in a compact subset of $\mathbb{D}$). What if the latter is not true?

2. In [41], Santos-León considers integrals of the form $\int_{-\pi}^{\pi} f(e^{i\theta})K(\theta)\mathrm{d}\theta$ with K such that $\int_{-\pi}^{\pi} |K(\theta)|\mathrm{d}\theta < \infty$. He proposes quadrature formulas of interpolatory type with nodes uniformly distributed on $\mathbb{T}$. Properties for the weights and estimates for the error of the quadrature formulas are given. A similar treatment can be given when the nodes are the zeros of the para-orthogonal rational functions with respect to the Lebesgue measure, which means the zeros of $z\omega_{n-1}(z) + \pi_{n-1}(z)$.

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