CONVERGENCE ACCELERATION FOR THE NUMERICAL SOLUTION OF SECOND-ORDER LINEAR RECURRENCE RELATIONS*

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Abstract. The concept of modification used for accelerating the convergence of ordinary continued fractions is adapted to the case of the Gautschi–Aggarwal–Burgmeier algorithm for the computation of nondominant solutions of nonhomogeneous second-order linear recurrence relations.

Key words. convergence acceleration, recurrence relation, continued fraction

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1. Introduction. One of the first algorithms for the calculation of nondominant solutions of linear recurrence relations was developed by Gautschi [4]–[6]. The algorithm is based on the relationship between recurrence relations and continued fractions. The method is only applicable in the case of a second-order linear homogeneous recurrence relation. In 1978 Aggarwal and Burgmeier [1] generalized this algorithm to the case of nonhomogeneous recurrence relations (see also Van der Cruyssen [23]). As we will show, this algorithm is closely related to Olver’s method of solving second-order recurrence relations [14], more specifically to Van der Cruyssen’s reformulation of Olver’s algorithm in terms of an LU-decomposition [22] (see also Cash [3], Lozier [13]).

In some cases the convergence of both of these algorithms is slow. The classical way to solve this problem is by using asymptotic information about the wanted solution of the recurrence relation (see, for instance, Gautschi [4], Scraton [19])—several authors have developed techniques to obtain the asymptotic behavior of all solutions of a linear recurrence relation; see Wimp [24]). The Gautschi algorithm (and its generalization to nonhomogeneous recurrence relations) lends itself very well to careful analysis when asymptotic values are used. It will be possible to prove convergence acceleration explicitly.

In § 2 we give a derivation of the generalized Gautschi algorithm and discuss its relationship with Olver’s algorithm [14], [22]. In § 3 we define a modification of this algorithm and in § 4 we show how this modification may be used to accelerate the convergence of the method. For homogeneous higher-order recurrence relations the same method has been used in [2], [10]–[12]. Finally, in § 5 we give some examples.

2. Definitions and notation. Consider the second-order linear nonhomogeneous recurrence relation

$$(1) \quad y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n = a_3(n), \quad n = 0, 1, \dots$$

with $a_2(n) \neq 0$. We assume that the homogeneous equation associated with (1):

$$(2) \quad y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n = 0, \quad n = 0, 1, \dots$$

has a minimal solution; this is a solution f_n such that for every other solution g_n of (2), linearly independent of f_n , we have

$$(3) \quad \lim_{n \rightarrow \infty} \frac{f_n}{g_n} = 0.$$

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(Let $f_0 \neq 0$.) Furthermore, we assume that the recurrence relation (1) has a solution h_n for which

$$(4) \quad \lim_{n \rightarrow \infty} \frac{h_n}{g_n} = 0$$

with g_n as defined above.

We denote by $A_n^{(1)}$ and $A_n^{(2)}$ the solutions of (2) with initial values:

$$\begin{aligned} A_0^{(1)} &= 1, & A_1^{(1)} &= 0, \\ A_0^{(2)} &= 0, & A_1^{(2)} &= 1 \end{aligned}$$

and by A_n the solution of (1) with

$$A_0 = 0, \quad A_1 = 0.$$

A solution p_n of (1) of the form

$$(5) \quad p_n = Cf_n + h_n, \quad n = 0, 1, \dots$$

for some constant $C \neq 0$ is uniquely defined by one initial condition. Forward recurrence with (1) is not a stable method to compute p_n . For the calculation of p_n we construct a first-order recurrence relation (Aggarwal and Burgmeier [1], Van der Cruyssen [23]):

$$(6) \quad y_{n+1} = b_1(n)y_n + b_2(n), \quad n = 0, 1, \dots,$$

which has a general solution of the form (5). To determine the coefficients in this recurrence relation we proceed as follows.

From (6) we get

$$(7) \quad \begin{aligned} y_{n+2} &= b_1(n+1)y_{n+1} + b_2(n+1) \\ &= b_1(n+1)(b_1(n)y_n + b_2(n)) + b_2(n+1). \end{aligned}$$

Substituting the expressions (6) and (7) into (1) gives us

$$((a_1(n) + b_1(n+1))b_1(n) + a_2(n))y_n = a_3(n) - b_2(n+1) - (a_1(n) + b_1(n+1))b_2(n).$$

Since every solution of (6) is a solution of (1) we must have

$$\begin{aligned} (a_1(n) + b_1(n+1))b_1(n) + a_2(n) &= 0, \\ a_3(n) - b_2(n+1) - (a_1(n) + b_1(n+1))b_2(n) &= 0. \end{aligned}$$

We rewrite these expressions as follows:

$$(8a) \quad b_1(n) = -\frac{a_2(n)}{a_1(n) + b_1(n+1)},$$

$$(8b) \quad b_2(n) = \frac{a_3(n) - b_2(n+1)}{a_1(n) + b_1(n+1)}.$$

Note that equation (8a) generates an ordinary continued fraction: in the case of a homogeneous recurrence relation the given method reduces to the continued fraction method of Gautschi [4]-[6]. As in the method of Gautschi approximations to the coefficients $b_1(n)$, $b_2(n)$ are obtained by calculating for an integer N :

$$(9) \quad \begin{aligned} b_{1,N}(N+1) &= b_{2,N}(N+1) = 0, \\ b_{1,N}(n) &= -\frac{a_2(n)}{a_1(n) + b_{1,N}(n+1)}, \\ b_{2,N}(n) &= \frac{a_3(n) - b_{2,N}(n+1)}{a_1(n) + b_{1,N}(n+1)}, \quad n = N, N-1, \dots, 0. \end{aligned}$$

If we want to calculate the solution p_n of (1) starting with the initial value p_0 , we get an approximation $p_n^{(N)}$ of p_n using

$$(10) \quad p_0^{(N)} = p_0, \quad p_{n+1}^{(N)} = b_{1,N}(n)p_n^{(N)} + b_{2,N}(n), \quad n = 0, 1, \dots, N,$$

in the sense that

$$(11) \quad \lim_{N \rightarrow \infty} p_n^{(N)} = p_n$$

for all values of n . The stability of this method has been proved in [10], [13].

Algorithm (9)–(10) is closely related to the algorithm first developed by Olver in [14] and reformulated in terms of an LU-decomposition by Van der Cruyssen [22]. To see this note that $p_n^{(N)}$ ($n = 0, \dots, N$) in (10) is a solution of the following system of equations:

$$(12) \quad \begin{aligned} z_0 &= p_0, \\ a_2(n)z_n + a_1(n)z_{n+1} + z_{n+2} &= a_3(n), \quad n = 0, \dots, N, \\ z_{N+2} &= 0, \end{aligned}$$

or

$$\begin{pmatrix} 1 & 0 & & & \\ a_2(0) & a_1(0) & 1 & & \\ & a_2(1) & a_1(1) & 1 & \\ & & & \ddots & \\ & & & & a_2(N) & a_1(N) & 1 \\ & & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_{N+1} \\ z_{N+2} \end{pmatrix} = \begin{pmatrix} p_0 \\ a_3(0) \\ a_3(1) \\ \vdots \\ a_3(N) \\ 0 \end{pmatrix}.$$

We decompose the matrix at the left-hand side in a product of an upper triangular and a lower triangular matrix:

$$\begin{pmatrix} 1 & 0 & & & \\ & 1 & \alpha(0) & & \\ & & 1 & \alpha(1) & \\ & & & \ddots & \\ & & & & 1 & \alpha(N) \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ a_2(0) & \beta(0) & & & \\ & a_2(1) & \beta(1) & & \\ & & \ddots & \ddots & \\ & & & a_2(N) & \beta(N) \\ & & & & 0 & 1 \end{pmatrix}$$

with

$$(13) \quad \begin{aligned} \alpha(N) &= 1, \quad \beta(N) = a_1(N), \\ \alpha(i-1) &= 1/\beta(i), \quad \beta(i-1) = a_1(i-1) - a_2(i)\alpha(i-1), \quad i = N, \dots, 1. \end{aligned}$$

Solving the system gives us

$$z_0 = p_0, \quad z_{i+1} = -\frac{a_2(i)}{\beta(i)} z_i + \frac{\gamma(i)}{\beta(i)}, \quad i = 0, \dots, N,$$

with $\gamma(n)$ given by

$$\gamma(N+1) = 0, \quad \gamma(i) = a_3(i) - \alpha(i)\gamma(i+1), \quad i = N, \dots, 0.$$

From these equations it is easy to deduce that

$$b_{1,N}(n) = -\frac{a_2(n)}{\beta(n)}, \quad b_{2,N}(n) = \frac{\gamma(n)}{\beta(n)}$$

for $n = 0, \dots, N$.

In the Olver-Van der Cruyssen method the system (12) is solved using an LU-decomposition (without pivoting). The method (9)-(10) is equivalent to using an LU-decomposition to solve the system

$$(14) \quad \begin{aligned} \dot{z}_0 &= 0, \\ \dot{z}_n + a_1(N-n)\dot{z}_{n+1} + a_2(N-n)\dot{z}_{n+2} &= a_3(N-n), \quad n = 0, \dots, N, \\ \dot{z}_{N+2} &= p_0 \end{aligned}$$

(i.e., the system (12) with the order of the equations reversed) and then putting

$$z_i = \dot{z}_{N+2-i}, \quad i = 0, \dots, N+2.$$

Remark. We note that algorithm (9)-(10) fails if one of the $\beta(i)$ in (13) is equal to zero. This only occurs if one of the following matrices:

$$\begin{pmatrix} a_1(i) & 1 & & & \\ a_2(i+1) & a_1(i+1) & 1 & & \\ & \dots & & \dots & \\ & & a_2(N) & a_1(N) & 1 \\ & & & 0 & 1 \end{pmatrix}, \quad i = 0, \dots, N$$

is singular (Horn and Johnson [7]). In most cases this problem can be solved by increasing N . For a detailed analysis, see Gautschi [4].

In the rest of this paper our main interest will lie in the calculation of p_1 given p_0 . Once p_1 is known, the same method, i.e., (10), may be applied to calculate p_2 from p_1 , and so on.

3. Modification and convergence of the algorithm. In algorithm (9) we have chosen the value zero as an approximation to $b_1(N+1)$ and $b_2(N+1)$. The choice is rather arbitrary and as we will see in the fourth paragraph modified approximants for $b_1(0)$ and $b_2(0)$ resulting from other choices than zero will often lead to acceleration of the convergence in (11). To introduce these modified approximants, we define transformations $s_n(u)$, $t_n(u, v)$, $S_n(u)$, and $T_n(u, v)$ by

$$(15) \quad \begin{aligned} s_n(u) &= -\frac{a_2(n)}{a_1(n) + u}, \quad t_n(u, v) = \frac{a_3(n) - v}{a_1(n) + u}, \quad n = 0, 1, \dots, \\ S_0(u) &= s_0(u), \quad T_0(u, v) = t_0(u, v), \\ S_n(u) &= S_{n-1}(s_n(u)), \quad T_n(u, v) = T_{n-1}(s_n(u), t_n(u, v)), \quad n = 1, 2, \dots \end{aligned}$$

Then it is easy to prove that our previous approximants resulted from setting $b_{1,N}(N+1) = u = 0$ and $b_{2,N}(N+1) = v = 0$:

$$b_{1,N}(0) = S_N(0), \quad b_{2,N}(0) = T_N(0, 0).$$

More generally, if

$$\begin{aligned} b_{1,N}(N+1) &= u, \quad b_{2,N}(N+1) = v, \\ b_{1,N}(n) &= -\frac{a_2(n)}{a_1(n) + b_{1,N}(n+1)}, \end{aligned}$$

$$b_{2,N}(n) = \frac{a_3(n) - b_{2,N}(n+1)}{a_1(n) + b_{1,N}(n+1)}, \quad n = N, N-1, \dots, 0,$$

then

$$b_{1,N}(0) = S_N(u), \quad b_{2,N}(0) = T_N(u, v).$$

Hence $S_n(u)$ and $T_n(u, v)$ are the approximations to $b_1(0)$ and $b_2(0)$ that we get if we replace $b_1(n+1)$ and $b_2(n+1)$ in (8) by u and v , respectively, and then use (8) to compute $b_1(0)$ and $b_2(0)$ backward. $S_n(u)$ and $T_n(u, v)$ are called *modified approximants* to $b_1(0)$ and $b_2(0)$.

We now have the following theorem.

THEOREM 1. For $k=0, 1, \dots$

$$(16a) \quad S_k(u) = -\frac{A_{k+2}^{(1)} - A_{k+1}^{(1)}u}{A_{k+2}^{(2)} - A_{k+1}^{(2)}u},$$

$$(16b) \quad T_k(u, v) = -\frac{A_{k+2} - A_{k+1}u - v}{A_{k+2}^{(2)} - A_{k+1}^{(2)}u}.$$

Proof. We use induction on k . For $k=0$ the right-hand side of equation (16b) is equal to

$$-\frac{A_2 - A_1u - v}{A_2^{(2)} - A_1^{(2)}u} = -\frac{a_3(0) - v}{-a_1(0) - u}$$

and this expression is equal to $T_0(u, v)$. For $k > 0$ let us assume that (16b) is true for $k = m-1$; then for $k = m$ we have

$$\begin{aligned} T_m(u, v) &= T_{m-1}(s_m(u), t_m(u, v)) \\ &= T_{m-1}\left(-\frac{a_2(m)}{a_1(m) + u}, \frac{a_3(m) - v}{a_1(m) + u}\right) \\ &= -\frac{A_{m+1}(a_1(m) + u) + A_m a_2(m) - a_3(m) + v}{A_{m+1}^{(2)}(a_1(m) + u) + A_m^{(2)} a_2(m)} \\ &= -\frac{a_1(m)A_{m+1} + a_2(m)A_m - a_3(m) + A_{m+1}u + v}{a_1(m)A_{m+1}^{(2)} + a_2(m)A_m^{(2)} + A_{m+1}^{(2)}u} \\ &= -\frac{-A_{m+2} + A_{m+1}u + v}{-A_{m+2}^{(2)} + A_{m+1}^{(2)}u} \end{aligned}$$

since A_n and $A_n^{(2)}$ are solutions of (1) and (2), respectively.

The same method may be used to prove (16a). $\square$

As a consequence of this theorem we have

$$(17) \quad S_n(0) = -\frac{A_{n+2}^{(1)}}{A_{n+2}^{(2)}}, \quad T_n(0, 0) = -\frac{A_{n+2}}{A_{n+2}^{(2)}}.$$

The method (9)–(10) is said to converge if the limits for $n \rightarrow \infty$ of the right-hand sides in (17) exist. If the conditions stated in the beginning of the previous paragraph ((3), (4), and $f_0 \neq 0$) are satisfied, then it is possible to prove convergence [10], [13]. Throughout this paper we assume that those conditions are satisfied.

4. Convergence acceleration. We want to calculate

$$p_1 = b_1(0)p_0 + b_2(0).$$

A first method to do this is given by (9) and (10). In some cases, however, the convergence of this algorithm will be slow. For a good choice of u and v in (15) the convergence will be accelerated.

Let us define

$$U_n(u, v) = p_0 S_n(u) + T_n(u, v)$$

with p_0 the initial value for the solution p_n of (1). We then have the following theorem.

THEOREM 2. For $k = 0, 1, \dots$ and for arbitrary u and v

$$(18a) \quad \frac{b_1(0) - S_k(u)}{b_1(0) - S_k(0)} = \left(1 - \frac{A_{k+1}^{(2)}}{A_{k+2}^{(2)}} u\right)^{-1} \left(1 - \frac{f_{k+1}}{f_{k+2}} u\right),$$

$$(18b) \quad \frac{b_2(0) - T_k(u, v)}{b_2(0) - T_k(0, 0)} = \left(1 - \frac{A_{k+1}^{(2)}}{A_{k+2}^{(2)}} u\right)^{-1} \left(1 - \frac{(\beta_1 f_{k+1} + h_{k+1})u + v}{\beta_1 f_{k+2} + h_{k+2}}\right),$$

$$(18c) \quad \frac{p_1 - U_k(u, v)}{p_1 - U_k(0, 0)} = \left(1 - \frac{A_{k+1}^{(2)}}{A_{k+2}^{(2)}} u\right)^{-1} \left(1 - \frac{p_{k+1}u + v}{p_{k+2}}\right)$$

with $\beta_1 = -h_0/f_0$.

Proof. We will prove (18c). There exist numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that

$$(19) \quad A_n = \alpha_1 A_n^{(2)} + \beta_1 f_n + h_n, \quad A_n^{(1)} = \alpha_2 A_n^{(2)} + \beta_2 f_n.$$

Since

$$\lim_{n \rightarrow \infty} \frac{A_n^{(1)}}{A_n^{(2)}} = -b_1(0), \quad \lim_{n \rightarrow \infty} \frac{A_n}{A_n^{(2)}} = -b_2(0),$$

we get for the numerator of the left-hand side of equation (18c):

$$\begin{aligned} p_1 - U_k(u, v) &= p_0(b_1(0) - S_k(u)) + b_2(0) - T_k(u, v) \\ &= \lim_{n \rightarrow \infty} \left(p_0 \left(-\frac{A_n^{(1)}}{A_n^{(2)}} - S_k(u) \right) - \frac{A_n}{A_n^{(2)}} - T_k(u, v) \right). \end{aligned}$$

Using Theorem 1 and (19), for the first term on the right we get

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left(-\frac{A_n^{(1)}}{A_n^{(2)}} - S_k(u) \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{A_{k+2}^{(1)} - A_{k+1}^{(1)}u}{A_{k+2}^{(2)} - A_{k+1}^{(2)}u} - \frac{A_n^{(1)}}{A_n^{(2)}} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{\alpha_2 A_{k+2}^{(2)} + \beta_2 f_{k+2} - (\alpha_2 A_{k+1}^{(2)} + \beta_2 f_{k+1})u}{A_{k+2}^{(2)} - A_{k+1}^{(2)}u} - \frac{\alpha_2 A_n^{(2)} + \beta_2 f_n}{A_n^{(2)}} \right) \\ &= \lim_{n \rightarrow \infty} \left(\frac{\beta_2 (f_{k+2} A_n^{(2)} - f_n A_{k+2}^{(2)} - (f_{k+1} A_n^{(2)} - f_n A_{k+1}^{(2)})u)}{A_n^{(2)} (A_{k+2}^{(2)} - A_{k+1}^{(2)}u)} \right) \\ &= \lim_{n \rightarrow \infty} \left(\beta_2 (A_{k+2}^{(2)} - A_{k+1}^{(2)}u)^{-1} \left(f_{k+2} - \frac{f_n}{A_n^{(2)}} A_{k+2}^{(2)} - \left(f_{k+1} - \frac{f_n}{A_n^{(2)}} A_{k+1}^{(2)} \right) u \right) \right) \\ &= (\beta_2 (A_{k+2}^{(2)} - A_{k+1}^{(2)}u)^{-1} (f_{k+2} - f_{k+1}u)) \end{aligned}$$

since

$$\lim_{n \rightarrow \infty} \frac{f_n}{A_n^{(2)}} = 0.$$

Using the same method we get for the second term

$$\lim_{n \rightarrow \infty} \left(-\frac{A_n}{A_n^{(2)}} - T_k(u, v) \right) = (A_{k+2}^{(2)} - A_{k+1}^{(2)}u)^{-1} (\beta_1 f_{k+2} + h_{k+2} - (\beta_1 f_{k+1} + h_{k+1})u - v)$$

and hence

$$p_1 - U_k(u, v) = (A_{k+2}^{(2)} - A_{k+1}^{(2)}u)^{-1}((\beta_2 p_0 + \beta_1)f_{k+2} + h_{k+2} - ((\beta_2 p_0 + \beta_1)f_{k+1} + h_{k+1})u - v).$$

From this it follows that

$$p_1 - U_k(0, 0) = (A_{k+2}^{(2)})^{-1}((\beta_2 p_0 + \beta_1)f_{k+2} + h_{k+2}).$$

From the initial conditions for $A_n^{(1)}$, $A_n^{(2)}$, and A_n it is easy to deduce that

$$\beta_1 = -\frac{h_0}{f_0}, \quad \beta_2 = \frac{1}{f_0},$$

and hence we have

$$(\beta_2 p_0 + \beta_1)f_n + h_n = \frac{p_0 - h_0}{f_0}f_n + h_n = p_n.$$

The last equality is true for all n since it is true for $n = 0$. We now have

$$\frac{p_1 - U_k(u, v)}{p_1 - U_k(0, 0)} = \left(\frac{A_{k+2}^{(2)} - A_{k+1}^{(2)}u}{A_{k+2}^{(2)}} \right)^{-1} \frac{p_{k+2} - p_{k+1}u - v}{p_{k+2}},$$

and this proves (18c). The same argument may be used to prove (18a) and (18b). $\square$

It is easy to see from (15) that

$$S_n(b_1(n+1)) = b_1(0), \quad T_n(b_1(n+1), b_2(n+1)) = b_2(0)$$

for all n . This is an extreme convergence acceleration. In general we do not know the exact values of $b_1(n+1)$ and $b_2(n+1)$ but in many cases it is possible to find approximations u_n and v_n to $b_1(n+1)$ and $b_2(n+1)$. Using these often leads to better approximations $S_n(u_n)$ and $T_n(u_n, v_n)$ to $b_1(0)$ and $b_2(0)$ than $S_n(0)$ and $T_n(0, 0)$.

In the following two cases we have convergence acceleration:

$$(20a) \quad \text{Case (i)} \quad \lim_{k \rightarrow \infty} \frac{b_1(0) - S_k(u_k)}{b_1(0) - S_k(0)} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{b_2(0) - T_k(u_k, v_k)}{b_2(0) - T_k(0, 0)} = 0;$$

$$(20b) \quad \text{Case (ii)} \quad \lim_{k \rightarrow \infty} \frac{p_1 - U_k(u_k, v_k)}{p_1 - U_k(0, 0)} = 0.$$

By using Theorem 2 it is easy to select values for u_k and v_k that will lead to convergence acceleration. We will always assume that the first factor in equations (18a)–(18c) satisfies

$$\left| 1 - \frac{A_{k+1}^{(2)}}{A_{k+2}^{(2)}} u_k \right| > M > 0$$

for some M and from some k on.

The classical method to accelerate the convergence of the algorithm (see, e.g., Piessens [18]) is solving the system (12) with $z_{N+2} = 0$ replaced by $z_{N+2} = p_{N+2}^{(\text{asym})}$ where $p_n \sim p_n^{(\text{asym})}$ ($n \rightarrow \infty$) (this means that $\lim_{n \rightarrow \infty} p_n^{(\text{asym})}/p_n = 1$). This is equivalent with the choice

$$(21) \quad u_N = 0, \quad v_N = p_{N+2}^{(\text{asym})}$$

as can be seen from (9)–(10).

It is not always easy to obtain expressions for the asymptotic behavior of the solutions of a recurrence relation although methods are available (Wimp [24]). The following theorem due to Perron [15] gives us some information about the asymptotic behavior of the solutions of a homogeneous linear recurrence relation.

THEOREM (Perron). *Let r be a natural number and let*

$$y_{n+r} + e_1(n)y_{n+r-1} + e_2(n)y_{n+r-2} + \cdots + e_r(n)y_n = 0$$

($n = 0, 1, \dots$) be a linear r th-order homogeneous recurrence relation, with

$$(22) \quad e_r(n) \neq 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} e_i(n) = e_i \quad (i = 1, \dots, r).$$

If the auxiliary equation

$$\rho^r + e_1\rho^{r-1} + e_2\rho^{r-2} + \cdots + e_r = 0$$

has roots $\rho_1, \dots, \rho_r$ all with distinct moduli, then the given recurrence relation has nontrivial solutions $f_n^{(j)}$ ($j = 1, \dots, r$) such that

$$(23) \quad \lim_{n \rightarrow \infty} \frac{f_{n+1}^{(j)}}{f_n^{(j)}} = \rho_j.$$

An extension of this theorem to homogeneous recurrence relations of a more general nature has been proved by Perron [16]–[17] and Kreuser [9].

In our case, to get the asymptotic behavior of the solution h_n of the non-homogeneous equation (1) we construct the third-order homogeneous recurrence relation that has the three linearly independent solutions f_n , g_n , and h_n . This can be done if $a_3(n) \neq 0$ for all n : from (1) we have

$$\frac{y_{n+2} + a_1(n)y_{n+1} + a_2(n)y_n}{a_3(n)} = 1 \quad \text{and} \quad \frac{y_{n+3} + a_1(n+1)y_{n+2} + a_2(n+1)y_{n+1}}{a_3(n+1)} = 1.$$

Since the right-hand sides of these equations are equal, the left-hand sides must also be equal, and after rearranging we find

$$\begin{aligned} y_{n+3} + \left(a_1(n+1) - \frac{a_3(n+1)}{a_3(n)} \right) y_{n+2} \\ + \left(a_2(n+1) - \frac{a_3(n+1)}{a_3(n)} a_1(n) \right) y_{n+1} - \frac{a_3(n+1)}{a_3(n)} a_2(n) y_n = 0. \end{aligned}$$

Then we use Perron's theorem or the methods described in [24] to determine the asymptotic behavior of h_n .

We look at some special cases.

Case (1). $\lim_{n \rightarrow \infty} h_n/f_n = \gamma$ for some number γ . From (5) we have $p_n \sim (C + \gamma)f_n$, ($n \rightarrow \infty$). If we take $u_k \sim f_{k+2}/f_{k+1}$ ($k \rightarrow \infty$) and $v_k = 0$, then we get convergence acceleration (Case (i)). To see this rewrite the second factor in (18b):

$$1 - \frac{(\beta_1 f_{k+1} + h_{k+1})u_k + v_k}{\beta_1 f_{k+2} + h_{k+2}} = 1 - \frac{\beta_1 + h_{k+1}/f_{k+1}}{\beta_1 + h_{k+2}/f_{k+2}} \frac{f_{k+1}}{f_{k+2}} u_k$$

and this proves (20a) under the condition that $\beta_1 + \gamma \neq 0$.

Case (2). $\lim_{n \rightarrow \infty} f_n/h_n = 0$. In this case we choose $u_k \sim h_{k+2}/h_{k+1}$ ($k \rightarrow \infty$) and $v_k = 0$. Since

$$\lim_{k \rightarrow \infty} \frac{p_{k+2}}{p_{k+1}} \frac{h_{k+2}}{h_{k+1}} = 1,$$

we get (20b).

Case (3). $a_3(n) = 0$, for all n ($\Rightarrow h_n \equiv 0$). If the given recurrence relation is homogeneous, then the problem is reduced to the calculation of the value of an ordinary continued fraction. Techniques for convergence acceleration of continued fractions may be found in [8] and [21] and the references given there.

Some remarks. (1) We note that when using the method described above to determine the asymptotic behavior of h_n (or p_n), we only get this up to a constant factor (since the third-order recurrence relation used is homogeneous). To determine this factor is not always easy. In the method (21) this factor is needed. However, if we use the modification described in Cases (1) and (2) it is easy to see that the factor cancels out.

(2) If the solution p_n is rapidly converging to zero, using the asymptotic expansion in (21) will cause problems when $|p_{N+2}^{(\text{asym})}|$ is smaller than $\varepsilon_{\min}$. This will not occur if we use the modifications from Cases (1) and (2). ($\varepsilon_{\min}$ is the smallest positive real number representable on the computer.)

5. Two examples.

Example 1. Consider the recurrence relation (1) with coefficients given by

$$a_1(n) = -2 - \frac{1}{(10n+11)^2},$$

$$a_2(n) = 0.99 + \frac{1}{(n+1)^3},$$

$$a_3(n) = 4 \exp(-1).$$

The asymptotic behavior of the solutions f_n and g_n of the associated homogeneous equation is given by

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = 0.9, \quad \lim_{n \rightarrow \infty} \frac{g_{n+1}}{g_n} = 1.1$$

and for the solution h_n we have

$$\lim_{n \rightarrow \infty} \frac{h_{n+1}}{h_n} = 1.$$

This is an example of Case (2). We have calculated an approximation to p_1 (with $p_0 = 1$) using four different choices for u_k and v_k (see Tables 1(a)–1(c)—calculations with 16 significant figures):

- (1) $u_k = 0$, $v_k = 0$, for all k (second column).
- (2) $u_k = 1$, $v_k = 0$, for all k (third column).
- (3) $u_k = 0$, $v_k = -400/e$, for all k (fourth column).
- (4) $u_k = 0.9$, $v_k = -40/e$, for all k (fifth column).

The first choice gives the original method (9)–(10), the second choice is the one described in the previous paragraph (Case (2)). If we want to use the classical method (21), we need more information about the solution p_n . Now, using a paper by Späth [20] it is possible to prove that for this example p_n satisfies

$$\lim_{n \rightarrow \infty} p_n = -\frac{400}{e}.$$

The third choice is given by (21) with $p_n^{(\text{asym})} = -400/e$.

As a consequence of Perron's theorem we have $\lim_{n \rightarrow \infty} b_1(n) = 0.9$. Since p_n is a solution of equation (6) we have

$$\lim_{n \rightarrow \infty} b_2(n) = \lim_{n \rightarrow \infty} p_{n+1} - \lim_{n \rightarrow \infty} b_1(n) \lim_{n \rightarrow \infty} p_n = -\frac{400}{e} - 0.9 \left(-\frac{400}{e} \right) = -\frac{40}{e}.$$

TABLE 1(a)
 $b_1(0) = 2.109233363$.

n	$S_n(0)$	$S_n(1)$	$S_n(0.9)$
50	2.109208819	2.109263364	2.109233361
60	2.109230063	2.109237397	2.109233363
100	2.109233362	2.109233365	2.109233363
110	2.109233363	2.109233364	2.109233363
120	2.109233363	2.109233363	2.109233363

TABLE 1(b)
 $b_2(0) = -18.74679823$.

n	$T_n(0, 0)$	$T_n(1, 0)$	$T_n(0, -400/e)$	$T_n(0.9, -40/e)$
90	-18.74081420	-18.74679804	-18.74679838	-18.74679822
100	-18.74449109	-18.74679821	-18.74679825	-18.74679823
110	-18.74590873	-18.74679823	-18.74679824	-18.74679823
120	-18.74645529	-18.74679823	-18.74679823	-18.74679823
250	-18.74679822	-18.74679823	-18.74679823	-18.74679823
260	-18.74679823	-18.74679823	-18.74679823	-18.74679823

TABLE 1(c)
 $p_1 = -16.63756486$.

n	$U_n(0, 0)$	$U_n(1, 0)$	$U_n(0, -400/e)$	$U_n(0.9, -40/e)$
100	-16.63525773	-16.63756485	-16.63756487	-16.63756487
110	-16.63667537	-16.63756486	-16.63756487	-16.63756487
120	-16.63722193	-16.63756486	-16.63756487	-16.63756487
130	-16.63743265	-16.63756486	-16.63756486	-16.63756486
230	-16.63756485	-16.63756486	-16.63756486	-16.63756486
240	-16.63756486	-16.63756486	-16.63756486	-16.63756486

By using this in Theorem 2 it is easy to prove that the fourth choice for u_k and v_k leads to convergence acceleration.

Note that the convergence of $S_n(1)$ is not as fast as that of $S_n(0)$ and $S_n(0.9)$. However, for the computation of p_1 the sequence $U_n(1, 0)$ converges faster than all the other methods, although less information was used about p_n than in the third and fourth choices for u_k and v_k .

Example 2. Consider the recurrence relation (1) with coefficients given by

$$a_1(n) = -4 \frac{n+2}{2n+3} - \frac{1}{50(n+1)^3},$$

$$a_2(n) = \frac{2n+5}{2n+3},$$

$$a_3(n) = 0.5^{n+1} \ln(n+2).$$

Then the asymptotic behavior of the solutions f_n and g_n is given by

$$\lim_{n \rightarrow \infty} f_n = c_1 \in \mathbb{R}, \quad \lim_{n \rightarrow \infty} \frac{g_n}{n^2} = c_2 \in \mathbb{R}$$

and for the solution h_n using Perron's theorem, we find

$$\lim_{n \rightarrow \infty} \frac{h_{n+1}}{h_n} = 0.5.$$

Numerical experiments show that

$$\lim_{n \rightarrow \infty} h_n = -0.42957 \dots$$

This is an example of Case (1). In Tables 2(a)–2(c) we have calculated an approximation to p_1 ($p_0 = 1$) using three different choices for u_k and v_k :

- (1) $u_k = 0, v_k = 0$, for all k (second column—see (9)–(10)).
- (2) $u_k = 1, v_k = 0$, for all k (third column—see case (2)).
- (3) $u_k = 0, v_k = -0.42957$, for all k (fourth column—see (21)).

We note that if the second term in the definition of $a_1(n)$ is omitted, then the resulting recurrence relation has the minimal solution $f_n = 1$ for all n .

TABLE 2(a)
 $b_1(0) = 0.9866939003$.

n	$S_n(0)$	$S_n(1)$
100	0.9864215590	0.9866939094
200	0.9866238022	0.9866939014
300	0.9866624382	0.9866939006
400	0.9866761154	0.9866939004
500	0.9866824842	0.9866939003
1000	0.9866910292	0.9866939003
5000	0.9866937849	0.9866939003

TABLE 2(b)
 $b_2(0) = -0.4288098297$.

n	$T_n(0, 0)$	$T_n(1, 0)$	$T_n(0, -0.42957)$
100	-0.4284182552	-0.4288098428	-0.4285374796
200	-0.4287090421	-0.4288098314	-0.4287397303
300	-0.4287645933	-0.4288098302	-0.4287783672
400	-0.4287842585	-0.4288098299	-0.4287920447
500	-0.4287934156	-0.4288098298	-0.4287984135
1000	-0.4288057017	-0.4288098297	-0.4288069587
5000	-0.4288096638	-0.4288098297	-0.4288097143

TABLE 2(c)
 $p_1 = 0.5578840705$.

n	$U_n(0, 0)$	$U_n(1, 0)$	$U_n(0, -0.42957)$
100	0.5580033038	0.5578840665	0.5578840794
200	0.5579147601	0.5578840700	0.5578840718
300	0.5578978449	0.5578840704	0.5578840710
400	0.5578918569	0.5578840705	0.5578840707
500	0.5578890686	0.5578840705	0.5578840706
1000	0.5578853275	0.5578840705	0.5578840706
5000	0.5578841211	0.5578840705	0.5578840705

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