DEPARTEMENT TOEGEPASTE ECONOMISCHE WETENSCHAPPEN

ONDERZOEKSRAPPORT NR 9627

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D/1996/2376/27

A STRAIGHTFORWARD ANALYTICAL CALCULATION OF THE DISTRIBUTION OF AN ANNUITY CERTAIN WITH STOCHASTIC INTEREST RATE[1]

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Abstract

Starting from the moment generating function of the annuity certain with stochastic interest rate written by means of a time discretization of the Wiener process as an n-fold integral, a straightforward evaluation of the corresponding distribution function is obtained letting n tend to infinity. The advantage of the present method consists in the direct calculation technique of the n-fold integral, instead of using moment calculation or differential equations, and in the possible applicability of the present method to varying annuities which could be applied to IBNR results, as well as to pension fund calculations, etc.

[1] Work performed under OT/93/5

1. Introduction

In the actuarial literature, the problem of how to incorporate the randomness of interest rates in the evaluation of actuarial quantities, has been given a lot of attention.

Different models for modelling the interest rates, as well as other types of rates, such as growth rates, inflation rates, etc., are studied. A lot of papers only obtain results for first and second order moment of the actuarial quantities under investigation. We mention e.g. the papers of Beeckman and Fuelling (1990,1991).

The papers by Yor (1992,1993), De Schepper et al (1992, 1994), give also results for the distribution of the annuity certain. Both authors make use of different techniques, to obtain the same result for the distribution. The randomness of the interest rates, is measured in both cases by a Wiener process.

The problem dealt with in the present paper is the straightforward evaluation of the generating function of the annuity certain with random interest rate

$$\bar{a}_{\overline{t|}R} = \int_0^t e^{-\delta\tau - x(\tau)} \, d\tau$$

where $x(\tau)$ represents a Brownian motion, and δ denotes the risk free interest intensity. This means that we aim at the direct evaluation of

$$E\left(e^{-u\bar{a}_{\overline{t|}R}} | x(0) = x_0\right)$$

by considering a discretization of the Wiener process with respect to the time variable, in n subintervals and subsequently letting n tend to infinity.
The expectation can be obtained by taking the integral over $x_t = x(t)$ of the expression

$$K(t,x(t);0,x(0)) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\varepsilon}^{\,n}} \int \cdots \int \prod_{j=1}^{n-1} dx_j . \exp\{-(\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2\varepsilon} + \varepsilon u e^{-\delta j\varepsilon - x_j})\}$$

as explained in De Schepper et al (1992).

The present method differs from the existing methods because straightforward evaluation of the (n-1)-fold intergral is presented instead of using differential equations or moment calculations.

The same methods will also enable us to obtain some limiting distributions of GARCH processes, as will be explained in a forthcoming paper.

2. Transformation of the (n-1)-fold Wiener integral

We start with the linear transformation of the variables $\delta j\varepsilon + x_j = y_j$ in the expression of K(t,x(t);0,x(0)). This results in :

$$K(t,x(t);0,x(0)) = e^{\frac{1}{2}\delta^2 t + \delta(x(t)-x(0))} K_0(t,x(t)+\delta t;0,x(0))$$

with

$$K_0(t,x(t)+\delta t;0,x(0)) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi\varepsilon}^{\,n}} \int\cdots\int \prod_{j=1}^{n-1} dy_j . \exp\{-(\sum_{j=1}^{n} \frac{(y_j-y_{j-1})^2}{2\varepsilon} + \varepsilon u e^{-y_j})\} \quad \textbf{(1)}$$

In the notation of Feynman integrals extended to imaginary time, the right hand side of this expression can be written as follows

$$K_0(t,y(t);0,y(0)) = \int_{(0,y(0))}^{(t,y(t))} Dy(\tau)\ \exp\{-\frac{1}{2}\int_0^t \dot{y}^2 d\tau - u\int_0^t e^{-y} d\tau\} \qquad \textbf{(2)}$$

where

$y(0) = x(0)$ *and* $y(t) = x(t) + \delta t$

The multiple integration in (1) is transformed by means of the coordinate transformation $y(\tau) = -\kappa \ln q(\tau)$ or rather in the discritized version of the coordinates $y_j = -\kappa \ln q_j$. For a general coordinate transformation the interested reader is referred to Khandekar and Lawande (1986).

Let $\Delta q_j = q_j - q_{j-1}$ and $\overline{q}_j = \dfrac{q_j + q_{j-1}}{2}$ then the following approximation for $\ln q_j$ in terms of Δq_j and $\overline{q}_j$ is obtained (preserving terms up to order ε^2) :

$$\ln q_j = \ln\overline{q}_j + \frac{\Delta q_j}{2}\frac{1}{\overline{q}_j} - \frac{\Delta q_j^2}{4}\frac{1}{2}\frac{1}{\overline{q}_j^{\,2}} + \frac{\Delta q_j^3}{8}\frac{1}{6}\frac{2}{\overline{q}_j^{\,3}} + O(\varepsilon^2)$$

from which an expression for part of the argument of the exponential in (1) can be given in terms of the new coordinate variables q_j

$$\frac{1}{2\varepsilon}(y_j - y_{j-1})^2 = \frac{\kappa^2}{2\varepsilon}\left(\frac{\Delta q_j}{\bar{q}_j} + \frac{(\Delta q_j)^3}{12}\frac{1}{\bar{q}_j^{\,3}}\right)^2 + O(\varepsilon^{3/2})$$

$$= \kappa^2 \frac{\Delta q_j^2}{2\varepsilon}\frac{1}{\bar{q}_j^{\,2}}\left(1 + \frac{(\Delta q_j)^2}{6}\frac{1}{\bar{q}_j^{\,2}}\right) + O(\varepsilon^{3/2})$$

In order to obtain a Wiener process, a new stochastic time is defined for the $q(\tau)$ process by means of

$$\varepsilon = \frac{\kappa}{q_j} \cdot \frac{\kappa}{q_{j-1}} \cdot \sigma_j + O(\varepsilon^{3/2}),$$

where σ_j denotes the new length of the time interval related to the new time variate t'. For symmetric reasons with respect to the indices the relation defining the new time is transformed as follows

$$\varepsilon = \frac{\kappa^2 \sigma_j}{\bar{q}_j^{\,2}}\left(1 + \frac{(\Delta q_j)^2}{4\bar{q}_j^{\,2}}\right) + O(\varepsilon^{3/2})$$

The continuous equivalent of this time relation gives, performing the integration of the time relation expression the following connection between the 'old time' t and the process depending 'new time' t' :

$$t = \int_0^{t'} \kappa^2 \frac{d\tau}{q^2(\tau)}$$

As a first result of this transformation the following equality (up to terms in $\varepsilon^{3/2}$) is obtained:

$$\sum_{j=1}^{n} \frac{1}{2\varepsilon}(y_j - y_{j-1})^2 = \sum_{j=1}^{n} \frac{(\Delta q_j)^2}{2\sigma_j}\left(1 - \frac{1}{12}\frac{(\Delta q_j)^2}{\bar{q}_j^{\,2}}\right) + O(\varepsilon^2)$$

as well as the transformed term

$$u \sum_{j=1}^{n} \varepsilon e^{-y_j} = u\kappa^2 \sum_{j=1}^{n} \bar{q}_j^{\,\kappa-2}\,\sigma_j + O(\varepsilon^{3/2})$$

In order to be able to express the (n-1) fold integral in (1) in terms of an (n-1) fold integral over the q-variates, we still have to evaluate the Jacobian of the transformation

$$\frac{1}{\sqrt{2\pi\varepsilon}^{\,n}} \prod_{j=1}^{n-1} dy_j = \prod_{j=1}^{n-1} \frac{\kappa dq_j}{q_j} \prod_{j=1}^{n} \frac{\sqrt{q_j q_{j-1}}}{\kappa\sqrt{2\pi\sigma_j}}(1 + O(\varepsilon))$$

$$= \sqrt{q_0 q_n}\,\frac{1}{\kappa} \prod_{j=1}^{n-1} dq_j \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_j}}(1 + O(\varepsilon))$$

Consequently the following intermediate result is obtained for the integrandum in the integral representation of K_0 (t,y(t) ; 0,y(0)) (before taking the limit)

$$e^{-\frac{y(0)}{2\kappa}} e^{-\frac{y(t)}{2\kappa}} \frac{1}{\kappa} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_j}} \prod_{j=1}^{n-1} dq_j$$

$$\exp\{-\sum_{j=1}^{n} \frac{(\Delta q_j)^2}{2\sigma_j} + \sum_{j=1}^{n} \frac{1}{24}\frac{(\Delta q_j)^4}{\bar{q}_j^{\,2}\sigma_j} - u\kappa^2 \sum_{j=1}^{n} \bar{q}_j^{\,\kappa-2}\sigma_j\}\ (1 + O(\varepsilon))$$

Because still an integration over the q_j 's has to be performed the second summation in the argument of the exponential is transformed using

$$\int_{-\infty}^{+\infty} k^4 e^{-\frac{k^2}{2\sigma}} dk = \frac{3}{4}(2\sigma)^2 \int_{-\infty}^{+\infty} e^{-\frac{k^2}{2\sigma}} dk$$

such that

$\int_{-\infty}^{+\infty} e^{-vk^4 - \frac{k^2}{2\sigma}} dk = \int_{-\infty}^{+\infty} e^{-3v\sigma^2 - \frac{k^2}{2\sigma}} dk + O(\sigma^4)$, as also explained in Khandekar ans Lawande (1986)

Consequently within the integrand of the (n-1) fold integral, taking into account $\bar{q}_j^{\,2} = \frac{\kappa^2\sigma_j}{\varepsilon}$,

the factor $\exp(\frac{1}{24}\frac{\Delta q_j^4}{\bar{q}_j^2\sigma_j})$ may be changed into the form $\exp(\frac{1}{8}\frac{\sigma_j}{\bar{q}_j^{\,2}})$.

One still has to connect the old time t to the new time t' by means of the relation $t = \int_0^{t'} \frac{\kappa^2 d\tau}{q^2(\tau)}$.
This can be done introducing a δ-function, and we make use of following relation

$$\delta\left[t - \int_0^{t'} d\tau \frac{\kappa^2}{q^2(\tau)}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\alpha(t - \int_0^{t'} d\tau \frac{\kappa^2}{q^2(\tau)})} d\alpha$$

Performing the additional integration over the stochastic time t', we obtain an integrandum of the following form

$$\int_{-\infty}^{+\infty} d\alpha\, e^{i\alpha t} \int_0^\infty dt' \; \frac{\kappa}{2\pi} e^{\frac{y(0)}{2\kappa}+\frac{y(t)}{2\kappa}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\sigma_j}}$$

$$\prod_{j=1}^{n-1} dq_j \exp\left\{ -\sum_{j=1}^{n} \left[\frac{(\Delta q_j)^2}{2\sigma_j} - \frac{1}{8}\frac{\sigma_j}{\bar{q}_j^{\,2}} \quad +i\alpha\kappa^2 \frac{\sigma_j}{\bar{q}_j^{\,2}} + u\kappa^2 \bar{q}_j^{\,\kappa-2}\sigma_j \right] \right\}$$

Integrating with respect to q_j 's, we obtain the following result in terms of the notation used in (2)

$$K_0(t,y(t);0,y(0)) = \frac{\kappa}{2\pi} \int_{-\infty}^{+\infty} d\alpha\, e^{i\alpha t}\, e^{\frac{y(0)}{2\kappa}+\frac{y(t)}{2\kappa}}$$

$$\int_0^\infty dt' \int_{(0,\exp(-y(0)/\kappa))}^{(t',\exp(-y(t)/\kappa))} D_+q(\tau)\; \exp\left\{ -\frac{1}{2}\int_0^{t'} \dot{q}^2 d\tau + \int_0^{t'} \frac{d\tau}{q^2}\left(\frac{1}{8} - \alpha i\kappa^2\right) - \kappa^2 u \int_0^{t'} q^{\kappa-2} d\tau \right\}$$

We use the notation $D_+q(\tau)$ to indicate that the integrations over q, only have to be performed over R^+.

Finally, performing the additional integration over x_t, we obtain the following result for the generating function of an annuity certain with the random interest is described as a Wiener process :

$$E[e^{-ua_{\bar{t}|R}} | x(0) = x_0] = \frac{\kappa}{\pi} e^{\delta^2\frac{t}{2}+\delta\frac{t}{2\kappa}} \int_{-\infty}^{\infty} dx_t\, e^{\delta(x_t - x_0)} \int_{-\infty}^{+\infty} d\alpha\, e^{i\alpha t} \int_0^\infty dt' e^{\frac{1}{2\kappa}x_0 + \frac{1}{2\kappa}x_t}$$

$$\int_{(0,\exp(-x_0/\kappa))}^{(t',\exp(-x_t/\kappa-\delta t/\kappa))} D_+q(\tau) \exp\{ -\frac{1}{2}\int_0^{t'} \dot{q}^2 d\tau + \int_0^{t'} \frac{d\tau}{q^2}\left(\frac{1}{8} - \kappa^2 i\alpha\right) - \kappa^2 u \int_0^{t'} q^{\kappa-2} d\tau \}$$

3. Analytical results

For special choices of the parameter value κ, some tractable integrations over q are obtained. Indeed, for the choice κ=4, a special case of theorem 1 of Vanneste et al is applicable. In this case, we find the following result :

$$M(u) = E[e^{-u a_{t\kappa}} | x(0) = x_0] = \frac{2}{\pi} e^{\delta^2 \frac{t}{2} + \delta \frac{t}{8}} \int_{-\infty}^{\infty} dx_t\, e^{\delta(x_t - x_0)} \int_{-\infty}^{+\infty} d\alpha\, e^{i\alpha t} \int_0^{\infty} dt'\, e^{\frac{1}{8}x_0 + \frac{1}{8}x_t}$$

$$\int_{(0,\exp(-x_0/4))}^{(t',\exp(-x_t/4-\delta t/4))} D_+ q(\tau) \exp\{-\frac{1}{2}\int_0^{t'} \dot{q}^2 d\tau + \int_0^{t'} \frac{d\tau}{q^2}(\frac{1}{8} - 16 i\alpha) - 16u \int_0^{t'} q^2 d\tau\}$$

Theorem 1 of Vanneste et al can be applied. In this case , we have

$$g = 16 i\alpha - \frac{1}{8}$$

and

$$\ddot{v}(\tau) = 32 v(\tau) \text{ giving } \eta(\tau) = \frac{\sinh(4\sqrt{2u}\,\tau)}{4\sqrt{2u}} \text{ and } \xi(\tau) = \cosh(4\sqrt{2u}\,\tau),$$

This gives the following result for the moment generating function :

$$M(u) = \frac{2}{\pi} e^{-\delta^2 \frac{t}{2}} \int_{-\infty}^{\infty} dz\, e^{\delta(z - x_0)} \int_{-\infty}^{\infty} d\alpha\, e^{i\alpha t} \int_0^{\infty} \frac{d\tau}{\tau\sqrt{\tau^2+1}}$$

$$\exp(-2\sqrt{2u}\,\frac{\sqrt{\tau^2+1}}{\tau}(e^{-\frac{x_0}{2}} + e^{-\frac{z}{2}}))\; I_{4\sqrt{2i\alpha}}(4\frac{\sqrt{2u}}{\tau} e^{-\frac{x_0}{4}-\frac{z}{4}})$$

with

$$z = x_t + \delta t \quad and \quad \tau = \sinh(4\sqrt{2u}\,t')$$

and $I_k(x)$ denotes as usual a modified Bessel function.

Transforming the Fourier transform with respect to α into an inverse Laplace transform with respect to s, the same original function is obtained in case the imaginary argument of the Fourier transform is considered as the variable of the image function in the Laplace transform. Hence, one obtains :

$$M(u) = 4e^{-\delta^2\frac{t}{2}} \int_{-\infty}^{\infty} dz\, e^{\delta(z-x_0)} \int_0^{\infty} \frac{d\tau}{\tau\sqrt{\tau^2+1}}$$

$$\exp(-2\sqrt{2u}\,\frac{\sqrt{\tau^2+1}}{\tau}(e^{-\frac{x_0}{2}}+e^{-\frac{z}{2}}))\ \mathscr{L}^{-1}\{I_{4\sqrt{2s}}(4\frac{\sqrt{2u}}{\tau}e^{-\frac{x_0}{4}-\frac{z}{4}});\ s\rightarrow t\}$$

The inverse Laplace transform with respect to s can be worked out based on a result given in De Schepper (1994) resulting in

$$M(u) = \frac{4\sqrt{u}}{\sqrt{\pi}\,\pi}\ \frac{1}{\sqrt{t}}\ e^{-\delta^2\frac{t}{2}}\ e^{\frac{8\pi^2}{t}} \int_{-\infty}^{\infty} dz\ e^{\delta(z-x_0)}\ e^{-\frac{x_0}{4}-\frac{z}{4}} \int_0^{\infty} d\tau\ \frac{1}{\tau^2\sqrt{\tau^2+1}}\ \exp\{-2\ \sqrt{2u}\ \frac{\sqrt{\tau^2+1}}{\tau}\ (e^{-\frac{x_0}{2}} + e^{-\frac{z}{2}})\}$$

$$\int_0^{\infty} dy\ e^{\frac{-8y^2}{t}}\ \sinh y\ \sin(\frac{16\pi y}{t})\ \exp\{-4\sqrt{2u}\ \frac{1}{\tau}e^{-\frac{x_0}{4}-\frac{z}{4}}\cosh y\}$$

One is still faced with one additional Laplace inversion with respect to u → x

$$\mathscr{L}^{-1}\{\sqrt{u}\,e^{-A\sqrt{u}}\ ;\ u\rightarrow x\} = \frac{1}{4\sqrt{\pi}}(A^2-2x)\ x^{-\frac{5}{2}}\ e^{-\frac{A^2}{4x}}$$

with

$$A = 2\sqrt{2}\,\frac{\sqrt{\tau^2+1}}{\sqrt{\tau}}\ (e^{-\frac{x_0}{2}} + e^{-\frac{z}{2}}) + 4\sqrt{2}\ \frac{1}{\tau}\ e^{-\frac{x_0}{4}-\frac{z}{4}}\ \cosh y$$

One finally obtains :

$$f_{a_{\eta R}}(x) = \frac{1}{\pi^2}\ \frac{1}{\sqrt{t}\ x^{\frac{5}{2}}}\, e^{-\delta^2\frac{t}{2}}\ e^{\frac{8\pi^2}{t}} \int_{-\infty}^{\infty} dz\ e^{\delta(z-x_0)}\ e^{-\frac{x_0}{4}-\frac{z}{4}}$$

$$\int_0^{\infty} d\tau\ \frac{1}{\tau^2\sqrt{\tau^2+1}} \int_0^{\infty} dy\ e^{\frac{-8y^2}{t}}\ \sinh y\ \sin(\frac{16\pi y}{t})\ (A^2-2x)\ e^{-\frac{A^2}{4x}}$$

On the other hand, if we take κ to be equal to 2, the complexity of the problem in our present setting diminish substantially. We have

$$M(u) = E[e^{-u a_{t\rceil R}} | x(0) = x_0] = \frac{1}{\pi} e^{\delta^2 \frac{t}{2} + \delta \frac{t}{4}} \int_{-\infty}^{\infty} dx_t \, e^{\delta(x_t - x_0)} \int_{-\infty}^{+\infty} d\alpha \, e^{i\alpha t} \int_0^{\infty} dt' e^{-4ut'} e^{\frac{1}{4}x_0 + \frac{1}{4}x_t}$$

$$\int_{(0,\exp(-x_0/2))}^{(t',\exp(-x_t/2 - \delta t/2))} D_+ q(\tau) \exp\{-\frac{1}{2}\int_0^{t'} \dot{q}^2 d\tau + \int_0^{t'} \frac{d\tau}{q^2}(\frac{1}{8} - 4i\alpha)\}$$

Theorem 1 of Vanneste et al can be applied with $\ddot{v}(\tau) = 0$ giving $\eta(\tau) = \tau$ and $\xi(\tau) = 1$, resulting in

$$M(u) = \frac{1}{\pi} e^{-\delta^2 \frac{t}{2}} \int_{-\infty}^{\infty} dz \, e^{\delta(z - x_0)} \int_{-\infty}^{\infty} d\alpha \, e^{i\alpha t} \int_0^{\infty} \frac{dx}{x} \, e^{-ux} \, e^{-\frac{2}{x}e^{-x_0} - \frac{2}{x}e^{-z}} I_{2\sqrt{2i\alpha}}(\frac{4}{x} e^{-\frac{x_0}{2} - \frac{z}{2}})$$

where

$$z = xt + \delta t \quad and \quad 4t' = x$$

In this case, we find immediately the inverse with respect to u. On the other hand, let $\mathscr{L}^{-1}$ denote the inverse Laplace transform, then we get (transforming the Fourier transform into a Laplace transform) :

$$f_{a_{t\rceil R}}(x) = \frac{2}{x} e^{-\delta^2 \frac{t}{2}} \int_{-\infty}^{\infty} dz \, e^{\delta(z - x_0)} \exp(-\frac{2}{x}e^{-x_0} - \frac{2}{x}e^{-z}) \underset{s \to t}{\mathscr{L}^{-1}} \{I_{2\sqrt{2s}}(\frac{4}{x} e^{-\frac{x_0}{2} - \frac{z}{2}})\}$$

from this we find the same result, as obtained by A. De Schepper (1994), using differential equations, and by M. Yor (1993) :

$$f_{a_{t\rceil R}}(x) = \frac{2\sqrt{2}}{\pi\sqrt{\pi}} \frac{1}{\sqrt{t}\, x^2} e^{-\delta^2 \frac{t}{2}} e^{\frac{2\pi^2}{t}} \int_{-\infty}^{\infty} dz e^{\delta(z - x_0)} e^{-\frac{x_0}{2} - \frac{z}{2}}$$

$$e^{-\frac{2}{x}e^{-x_0} - \frac{2}{x}e^{-z}} \int_0^{\infty} dy \, e^{-2\frac{y^2}{t}} e^{-\frac{4}{x}e^{-\frac{x_0}{2} - \frac{z}{2}} \cosh y} \sinh y \, \sin(\frac{4\pi y}{t})$$

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