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Some Results on Moments and Cumulants

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Abstract

In the present paper we discuss various results related to moments and cumulants of probability distributions and approximations to probability distributions. As the approximations are not necessarily probability distributions themselves, we shall apply the concept of moments and cumulants to more general functions. Recursions are deduced for the moments and cumulants of functions in the form $R_k[\mathbf{a},\mathbf{b}]$ as defined by Dhaene & Sundt (1994). We deduce a simple relation between the De Pril transform and the cumulants of a function. This relation is applied to some classes of approximations to probability distributions, in particular the approximations of Hipp and De Pril.

Keywords: Moments, cumulants, De Pril transforms, recursions, approximations.

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1. Introduction

1A. In the present paper we discuss various results related to moments and cumulants of probability distributions and approximations to probability distributions.

In Section 2 we give some introductory remarks on moments and cumulants. As we are going to discuss approximations to probability distributions by functions which are not necessarily probability distributions themselves, we extend the definition of moments and cumulants to more general functions.

In Section 3 we discuss recursions for moments of functions f on the non-negative integers that satisfy a recursion in the form

$$f(x) = \Sigma_{y=1}^{k} \left[a_y + \frac{b_y}{x}\right] f(x-y) \qquad (x=1,2,...) \qquad (1.1)$$

with $f(0)>0$ and $f(x)=0$ for $x<0$; we allow $k=\infty$. Probability distributions that satisfy a recursion in the form (1.1) were studied by Sundt (1992), and the analysis was extended to more general functions by Dhaene & Sundt (1994).

It is easily seen that every function f on the non-negative integers with $f(0)>0$ satisfies a recursion in the form (1.1) with $k=\infty$ and $a_y=0$ for all y. The b_y's are uniquely determined by f. We call the function φ_f defined by $\varphi_f(0)=0$ and $\varphi_f(x)=b_x$ $(x=1,2,...)$ the *De Pril transform* of f. The De Pril transform was defined for probability distributions by Sundt (1995), motivated by De Pril (1989) and Dhaene & De Pril (1994), and the definition was extended to more general functions by Dhaene & Sundt (1994). In Section 4 we shall deduce a relation between the cumulants and the De Pril transform of f. As an application we deduce a recursion for the cumulants of a function f that satisfies the recursion (1.1).

In Section 5 we apply the results of Section 4 to some classes of approximations to probability distributions. In particular we discuss the approximations of Hipp (1986) and De Pril (1989).

1B. In the present paper we shall identify a probability distribution on the integers by its discrete density. For convenience we shall therefore usually mean the discrete density when referring to a distribution.

We denote by I the indicator function defined by $I(A)=1$ if the condition A is true and $I(A)=0$ if it is false. Furthermore, we shall interpret $\Sigma_{i=a}^{b} v_i = 0$ if $a>b$.

2. Moments and cumulants

2A. Let $\mathcal{P}$ denote the class of probability distributions on the non-negative integers. We shall denote the jth order moment of $f\in\mathcal{P}$ by $\mu_j(f)$, the jth order cumulant by $\kappa_j(f)$, the moment generating function by τ_f, and the cumulant generating function by θ_f, that is,

$$\mu_j(f) = \Sigma_{x=0}^{\infty} x^j f(x) \qquad (j=0,1,\ldots) \tag{2.1}$$

$$\tau_f(s) = \Sigma_{x=0}^{\infty} e^{sx} f(x) \tag{2.2}$$

$$\theta_f(s) = \ln \tau_f(s) \tag{2.3}$$

$$\kappa_j(f) = \frac{\mathrm{d}^j}{\mathrm{d}s^j}\,\theta_f(s)\Big|_{s=0}. \qquad (j=0,1,\ldots) \tag{2.4}$$

In particular we have $\mu_0(f)=1$ and $\kappa_0(f)=0$.

The moments can be obtained from the moment generating function by

$$\mu_j(f) = \frac{d^j}{ds^j}\tau_f(s)\Big|_{s=0} \qquad (j=0,1,...) \qquad (2.5)$$

and from the cumulants by the recursion

$$\mu_j(f) = \Sigma_{i=1}^{j} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix} \kappa_i(f)\mu_{j-i}(f), \qquad (j=1,2,...) \qquad (2.6)$$

which is obtained by (2.4), (2.5), and

$$\tau'_f(s) = \theta'_f(s)\tau_f(s). \qquad (2.7)$$

By solving (2.6) with respect to $\kappa_j(f)$ we obtain

$$\kappa_j(f) = \frac{1}{\mu_0(f)}\left[\mu_j(f) - \Sigma_{i=1}^{j-1} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix} \kappa_i(f)\mu_{j-i}(f)\right]; \qquad (j=1,2,...) \qquad (2.8)$$

for $f \in \mathcal{P}$ we have $\mu_0(f)=1$. From (2.6) and (2.8) we see that for any positive integer r, there is a one-to-one relation between the moments of orders $1,2,...,r$ and the corresponding cumulants.

2B. As we are going to discuss approximations to probability distributions by functions which are not necessarily probability distributions themselves, we shall now extend the definition of moments and cumulants to more general functions.

Let $\mathcal{F}$ denote the class of functions on the non-negative integers. The definition (2.1) of moments is easily extended to functions $f \in \mathcal{F}$ when the summations exist. As a function in $\mathcal{F}$ does not necessarily sum to one like a probability distribution, the zeroth order moment becomes more interesting.

If we analogously extend the definition of cumulants by (2.2)–(2.4) to functions $f\in\mathcal{F}$, then existence become more problematic. We see that if $\mu_0(f)<0$, then θ_f does not exist in a neighbourhood around zero. This implies that even when the moments of orders $1,2,...,r$ exist, the corresponding cumulants do not necessarily exist. This problem could be avoided by e.g. defining the cumulants by (2.8). However, when we discuss cumulants in later sections, we will have $\mu_0(f)>0$ so that we can stick to the definition by (2.2)–(2.4). The relations (2.5)–(2.8) still hold under this generalisation.

Let $f\in\mathcal{F}$ and c be a positive constant. Then

$$\tau_{cf}(s) = c\tau_f(s) \qquad \theta_{cf}(s) = \ln c + \theta_f(s)$$
$$\mu_j(cf) = c\mu_j(f) \qquad (j=0,1,...)$$
$$\kappa_j(cf) = I(j=0) \ln c + \kappa_j(f) \,. \qquad (j=0,1,...) \tag{2.10}$$

From (2.10) we see that all cumulants except the one of order zero are invariant against scale transforms of the function. We notice that the zeroth order cumulant does not appear in the recursion (2.6). However, the correct scaling of the moments is ensured by the initial value $\mu_0(f)$.

3. Functions in the form $R_k[\mathbf{a},\mathbf{b}]$

3A. Let $\mathcal{F}_0$ denote the class of all functions on the non-negative integers with a positive mass in zero and $\mathcal{P}_0$ the class of probability distributions in $\mathcal{F}_0$. Sundt (1992) denoted by $R_k[\mathbf{a},\mathbf{b}]$ the distribution $f\in\mathcal{P}_0$ defined by the recursion (1.1) with $\mathbf{a}=(a_1,...,a_k)$ and $\mathbf{b}=(b_1,...,b_k)$. More generally, Dhaene & Sundt (1994) defined a function $f\in\mathcal{F}_0$ to be in the form $R_k[\mathbf{a},\mathbf{b}]$ if it satisfies the recursion (1.1). In the following we shall allow k to be finite or infinite unless stated otherwise.

When analysing functions in the form $R_k[\mathbf{a},\mathbf{b}]$ we shall sometimes for convenience silently apply $a_y=b_y=0$ for $y>k$ and $y\leq 0$.

3B. Let $\mathcal{F}_+$ denote the class of functions on the positive integers and $\mathcal{P}_+$ the class of distributions in $\mathcal{F}_+$. We shall now consider compound functions in the form $p\vee h$ defined by

$$(p\vee h)(x) = \Sigma_{n=0}^{x} p(n)h^{n*}(x) \qquad (x=0,1,2,...) \tag{3.1}$$

with $h\in\mathcal{F}_+$ and $p\in\mathcal{F}_0$ in the form $R_k[\mathbf{a},\mathbf{b}]$.

The following theorem was proved by Sundt (1992) in the special case when $p\in\mathcal{P}_0$ and $h\in\mathcal{P}_+$.

Theorem 3.1. *If $h\in\mathcal{F}_+$ and $p\in\mathcal{F}_0$ is in the form $R_k[\mathbf{a},\mathbf{b}]$, then*

$$(p\vee h)(x) = \Sigma_{y=1}^{x} (p\vee h)(x-y)\, \Sigma_{i=1}^{k} \left[a_i + \frac{b_i}{i}\frac{y}{x}\right] h^{i*}(y) \qquad (x=1,2,...) \tag{3.2}$$

$$(p\vee h)(0) = p(0). \tag{3.3}$$

Proof. Formula (3.3) follows immediately from the definition (3.1).

Now let x be a positive integer. Then

$$(p\vee h)(x) = \Sigma_{n=1}^{x} p(n)h^{n*}(x) = \Sigma_{n=1}^{x} \Sigma_{i=1}^{n} \left[a_i + \frac{b_i}{n}\right] p(n-i)h^{n*}(x) =$$
$$\Sigma_{i=1}^{x} \Sigma_{n=i}^{x} \left[a_i + \frac{b_i}{n}\right] p(n-i)h^{n*}(x) = \Sigma_{i=1}^{x} \Sigma_{n=0}^{x} \left[a_i + \frac{b_i}{n+i}\right] p(n)h^{(n+i)*}(x).$$

From Lemma 6.1 in Dhaene & Sundt (1994) we obtain

$$h^{(n+i)*}(x) = \frac{1}{x}\frac{n+i}{i}\Sigma_{y=1}^{x}\, yh^{i*}(y)h^{n*}(x-y).$$

Thus

$$(p\vee h)(x) = \Sigma_{i=1}^{x}\,\Sigma_{n=0}^{x}\,\Sigma_{y=1}^{x}\left[a_i + \frac{b_i}{i}\frac{y}{x}\right]p(n)h^{i*}(y)h^{n*}(x-y) =$$
$$\Sigma_{y=1}^{x}\,\Sigma_{i=1}^{x}\left[a_i + \frac{b_i}{i}\frac{y}{x}\right]h^{i*}(y)\,\Sigma_{n=0}^{x}\,p(n)h^{n*}(x-y) =$$
$$\Sigma_{y=1}^{x}\,\Sigma_{i=1}^{k}\left[a_i + \frac{b_i}{i}\frac{y}{x}\right]h^{i*}(y)(p\vee h)(x-y),$$

which proves (3.2).

This completes the proof of Theorem 3.1. Q.E.D.

We immediately obtain the following corollary to Theorem 3.1.

Corollary 3.1. *If* $h\in\mathcal{F}_+$ *and* $p\in\mathcal{F}_0$ *is in the form* $R_k[\mathbf{a},\mathbf{b}]$*, then* $p\vee h$ *is in the form* $R_m[\mathbf{c},\mathbf{d}]$ *with*

$$m = k\sup\{y\colon h(y)>0\}$$
$$c_x = \Sigma_{y=1}^{k}\, a_y h^{y*}(x) \qquad (x=1,\ldots,m) \qquad (3.4)$$
$$d_x = x\,\Sigma_{y=1}^{k}\,\frac{b_y}{y}\,h^{y*}(x). \qquad (x=1,\ldots,m) \qquad (3.5)$$

3C. Before continuing with the general case, we shall in this subsection consider the special case $k=1$. For a function in the form $R_1[a,b]$, the recursion (3.2)–(3.3) reduces to

$$(p\vee h)(x) = \Sigma_{y=1}^{x}\left[a + b\,\frac{y}{x}\right]h(y)(p\vee h)(x-y) \qquad (x=1,2,\ldots) \qquad (3.6)$$
$$(p\vee h)(0) = p(0). \qquad (3.7)$$

This recursion was deduced by Panjer (1981) for the case when $p \in \mathcal{P}_0$ and $h \in \mathcal{P}_+$.

Sundt & Jewell (1981) showed that for the distribution $R_1[a,b]$ we always have $a<1$, and that this distribution is binomial if $a<0$, Poisson if $a=0$, and negative binomial if $0<a<1$. As we shall need the binomial and negative binomial distributions later, we shall display the recursion (3.6)–(3.7) for each of these two cases.

i) *Binomial.*

$$p(x) = \binom{t}{x} \pi^x (1-\pi)^{t-x}. \qquad (x=0,1\dots,t;\ t=1,2,\dots;\ 0<\pi<1) \qquad (3.8)$$

Then

$$a = -\frac{\pi}{1-\pi} \qquad b = (t+1)\frac{\pi}{1-\pi}, \qquad (3.9)$$

and we obtain

$$(p \vee h)(x) = \frac{\pi}{1-\pi} \Sigma_{y=1}^{x} \left[(t+1)\frac{y}{x} - 1\right] h(y)(p \vee h)(x-y) \qquad (x=1,2,\dots)$$
$$(p \vee h)(0) = (1-\pi)^t.$$

ii) *Negative binomial.*

$$p(x) = \binom{\alpha+x-1}{x} (1-\pi)^{\alpha} \pi^x. \qquad (x=0,1,\dots;\ \alpha>0;\ 0<\pi<1) \qquad (3.10)$$

Then

$$a = \pi \qquad b = (\alpha-1)\pi,$$

and we obtain

$$(p\vee h)(x) = \pi\, \Sigma_{y=1}^{x} \left[1 + (\alpha-1)\frac{y}{x}\right] h(y)(p\vee h)(x-y) \qquad (x=1,2,\ldots) \qquad (3.11)$$
$$(p\vee h)(0) = (1-\pi)^{\alpha}.$$

3D. In this subsection we shall consider moments of functions in the form $R_k[\mathbf{a},\mathbf{b}]$.

For a vector $\mathbf{v}= (v_1,..,v_m)$ we introduce

$$|\mathbf{v}| = (|v_1|,..,|v_m|)$$
$$v_x^+ = \max(v_x,0) \qquad (x=1,2,\ldots,m)$$
$$\mathbf{v}^+ = (v_1^+,\ldots,v_m^+)$$

and analogous to our notation for moments of a function

$$\mu_j(\mathbf{v}) = \Sigma_{x=1}^{m} x^j v_x. \qquad (j=-1,0,1,\ldots)$$

The following lemma gives sufficient conditions for the moments of a function in the form $R_k[\mathbf{a},\mathbf{b}]$ to exist.

Lemma 3.1. *Let $f\in\mathcal{F}_0$ be in the form $R_k[\mathbf{a},\mathbf{b}]$ and n be a non-negative integer. If*

$$\mu_0(|\mathbf{a}|) < 1 \qquad (3.12)$$
$$\mu_n(|\mathbf{a}|) < \infty \qquad (3.13)$$
$$\mu_{n-1}(|\mathbf{b}|) < \infty, \qquad (3.14)$$

then $\mu_n(|f|)<\infty$.

Proof. If $\mathbf{a}=\mathbf{b}=\mathbf{0}$, the lemma trivially holds. We therefore consider the complementary case.

For some

$$\alpha > 1 + \frac{\mu_{-1}(|\mathbf{b}|)}{1-\mu_0(|\mathbf{a}|)}, \tag{3.15}$$

let

$$\pi = \mu_0(|\mathbf{a}|) + \frac{1}{\alpha-1}\,\mu_{-1}(|\mathbf{b}|) \tag{3.16}$$

$$h(y) = \frac{1}{\pi}\left[|a_y| + \frac{|b_y|}{y(\alpha-1)}\right]. \qquad (y=1,2,\dots,k) \tag{3.17}$$

Then $h\in\mathcal{P}_+$, and from (3.12) and (3.15) we see that $0<\pi<1$.

Let g be a compound negative binomial distribution with severity distribution h and counting distribution given by (3.10) with α and π given by (3.15) and (3.16). From (3.17), (3.13), and (3.14) we obtain

$$\mu_n(h) = \frac{1}{\pi}\left[\mu_n(|\mathbf{a}|) + \frac{1}{\alpha-1}\,\mu_{n-1}(|\mathbf{b}|)\right] < \infty,$$

and thus

$$\mu_n(g) < \infty. \tag{3.18}$$

We shall now prove by induction that

$$|f(x)| \le \frac{f(0)}{g(0)}\, g(x). \qquad (x=0,1,...) \qquad (3.19)$$

It is immediately seen that (3.19) holds for $x=0$. Let us now assume that it holds for $x=0,1,...,z$. By using in turn (1.1), (3.19), (3.16), (3.17), and (3.11) we obtain

$$|f(z)| = \left|\Sigma_{y=1}^{k}\left[a_y + \frac{b_y}{z}\right]f(z-y)\right| \le \Sigma_{y=1}^{k}\left[|a_y| + \frac{|b_y|}{z}\right]|f(z-y)| \le$$
$$\frac{f(0)}{g(0)}\Sigma_{y=1}^{k}\left[|a_y| + \frac{|b_y|}{z}\right]g(z-y) \le \frac{f(0)}{g(0)}\Sigma_{y=1}^{k}\left[|a_y| + \frac{|b_y|}{y(\alpha-1)}\right]\left[1+(\alpha-1)\frac{y}{z}\right]g(z-y) =$$
$$\frac{f(0)}{g(0)}\,\pi\,\Sigma_{y=1}^{k}\left[1+(\alpha-1)\frac{y}{z}\right]h(y)g(z-y) = \frac{f(0)}{g(0)}\, g(z).$$

Thus (3.19) holds for $x=0,1,...$

From (3.18) and (3.19) we finally obtain

$$\mu_n(|f|) \le \frac{f(0)}{g(0)}\,\mu_n(g) < \infty. \qquad \text{Q.E.D.}$$

From the following lemma we see that if $f\in\mathcal{P}_0$, then we can relax the assumptions (3.12)–(3.14).

Lemma 3.2. *Let f be the distribution $R_k[\mathbf{a},\mathbf{b}]$ and n be a non-negative integer. If the inequalities*

$$\mu_0(\mathbf{a}^+)<1, \qquad (3.20)$$

(3.13), *and* (3.14) *are satisfied, then $\mu_n(f)<\infty$.*

Proof. Let $g\in\mathcal{F}_0$ be in the form $R_k[\mathbf{a}^+,\mathbf{b}^+]$ with $g(0)=f(0)$. Then Lemma 3.1 gives that $\mu_n(g)<\infty$. Utilising that f is non-negative, it is easily proved by induction that $f(x) \le g(x)$ for $x=1,2,\ldots$ Thus $\mu_n(f) \le \mu_n(g) < \infty$. Q.E.D.

If $f \in \mathcal{F}_0$ is in the form $R_k[\mathbf{a},\mathbf{b}]$ with k finite, then (3.13) and (3.14) are satisfied for all non-negative integers n, so that in this case the condition (3.12), or (3.20) if $f \in \mathcal{P}_0$, is a sufficient condition for the existence of moments of f of all orders.

Theorem 3.2. *Let $f \in \mathcal{F}_0$ be in the form $R_k[\mathbf{a},\mathbf{b}]$ and n a positive integer. If* (3.12) *(or* (3.20) *if $f \in \mathcal{P}_0$),* (3.13), *and* (3.14) *are satisfied, then the moments of f of order $j=1,2,\ldots,n$ exist, are finite, and satisfy the recursion*

$$\mu_j(f) = \frac{1}{1-\mu_0(\mathbf{a})} \sum_{i=1}^{j} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix} \left[\frac{j}{i} \mu_i(\mathbf{a}) + \mu_{i-1}(\mathbf{b}) \right] \mu_{j-i}(f). \tag{3.21}$$

Proof. For $j=1,.2,\ldots,n$ we have

$$\begin{aligned}
\mu_j(f) &= \sum_{x=1}^{\infty} x^j f(x) = \sum_{x=1}^{\infty} x^j \sum_{y=1}^{x} \left[a_y + \frac{b_y}{x} \right] f(x-y) = \\
&\sum_{y=1}^{\infty} a_y \sum_{x=y}^{\infty} x^j f(x-y) + \sum_{y=1}^{\infty} b_y \sum_{x=y}^{\infty} x^{j-1} f(x-y) = \\
&\sum_{y=1}^{\infty} a_y \sum_{x=0}^{\infty} (x+y)^j f(x) + \sum_{y=1}^{\infty} b_y \sum_{x=0}^{\infty} (x+y)^{j-1} f(x) = \\
&\sum_{y=1}^{\infty} a_y \sum_{x=0}^{\infty} \sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} y^i x^{j-i} f(x) + \sum_{y=1}^{\infty} b_y \sum_{x=0}^{\infty} \sum_{i=0}^{j-1} \begin{bmatrix} j-1 \\ i \end{bmatrix} y^i x^{j-1-i} f(x) = \\
&\sum_{i=0}^{j} \begin{bmatrix} j \\ i \end{bmatrix} \mu_i(\mathbf{a}) \mu_{j-i}(f) + \sum_{i=0}^{j-1} \begin{bmatrix} j-1 \\ i \end{bmatrix} \mu_i(\mathbf{b}) \mu_{j-i-1}(f) = \\
&\mu_0(\mathbf{a}) \mu_j(f) + \sum_{i=1}^{j} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix} \left[\frac{j}{i} \mu_i(\mathbf{a}) + \mu_{i-1}(\mathbf{b}) \right] \mu_{j-i}(f),
\end{aligned}$$

from which we obtain (3.21). Q.E.D.

Theorem 3.3. *Let $p \in \mathcal{F}_0$ be in the form $R_k[\mathbf{a},\mathbf{b}]$ with $k<\infty$, $h \in \mathcal{P}_+$, and n a positive integer. If $\mu_0(|\mathbf{a}|)<1$ (or $\mu_0(\mathbf{a}^+)<1$ if $f \in \mathcal{P}_0$) and $\mu_n(h)<\infty$, then the moments of $p \vee h$ of order $j=1,2,\ldots,n$ exist, are finite, and satisfy the recursion*

$$\mu_j(p\vee h) = \frac{1}{1-\mu_0(\mathbf{a})}\,\Sigma_{i=1}^{j}\,\binom{j-1}{i-1}\left[\tfrac{j}{i}\,\mu_i(\mathbf{c}) + \mu_{i-1}(\mathbf{d})\right]\mu_{j-i}(p\vee h) \tag{3.22}$$

with

$$\mu_i(\mathbf{c}) = \Sigma_{y=1}^{k}\, a_y\mu_i(h^{y*}) \qquad \mu_i(\mathbf{d}) = \Sigma_{y=1}^{k}\,\frac{b_y}{y}\,\mu_{i+1}(h^{y*}). \qquad (i=0,1,\dots,n)$$

Proof. By insertion of **c** and **d** defined by (3.4) and (3.5) in (3.21) we obtain the recursion (3.22) with

$$\mu_i(\mathbf{c}) = \Sigma_{x=1}^{\infty}\, x^i\,\Sigma_{y=1}^{k}\, a_y h^{y*}(x) = \Sigma_{y=1}^{k}\, a_y\mu_i(h^{y*})$$
$$\mu_i(\mathbf{d}) = \Sigma_{x=1}^{\infty}\, x^{i+1}\,\Sigma_{y=1}^{k}\,\frac{b_y}{y}\,h^{y*}(x) = \Sigma_{y=1}^{k}\,\frac{b_y}{y}\,\mu_{i+1}(h^{y*}).$$

so that the theorem is proved if we can show that $\mu_0(|\mathbf{c}|)<1$ (or $\mu_0(\mathbf{c}^+)<1$ if $f\in\mathcal{P}_0$ and $\mu_0(\mathbf{a}^+)<1$), $\mu_n(|\mathbf{c}|)<\infty$, and $\mu_{n-1}(|\mathbf{d}|)<\infty$. We have

$$\mu_0(|\mathbf{c}|) = \Sigma_{x=1}^{\infty}\,|c_x| \le \Sigma_{x=1}^{\infty}\,\Sigma_{y=1}^{k}\,|a_y|\,h^{y*}(x) = \Sigma_{y=1}^{k}\,|a_y| < 1;$$

if $\mu_0(\mathbf{a}^+)<1$, we analogously show that $\mu_0(\mathbf{c}^+)<1$. Furthermore,

$$\mu_n(|\mathbf{c}|) = \Sigma_{x=1}^{\infty}\, x^n|c_x| \le \Sigma_{x=1}^{\infty}\, x^n\,\Sigma_{y=1}^{k}\,|a_y|\,h^{y*}(x) = \Sigma_{y=1}^{k}\,|a_y|\,\mu_n(h^{y*}) < \infty$$

$$\mu_{n-1}(|\mathbf{d}|) = \Sigma_{x=1}^{\infty}\, x^{n-1}|d_x| \le \Sigma_{x=1}^{\infty}\, x^n\,\Sigma_{y=1}^{k}\,\frac{|b_y|}{y}\,h^{y*}(x) =$$
$$\Sigma_{y=1}^{k}\,\frac{|b_y|}{y}\,\mu_n(h^{y*}) < \infty.$$

This completes the proof of Theorem 3.3. Q.E.D.

Theorem 3.3 could have been stated more generally, but then the regularity conditions would have become more cumbersome.

Let us now return to the special case $k=1$. From Lemma 3.1 we see that the moments of all orders of a function in the form $R_1[a,b]$ exist and are finite if $|a|<1$, and from Lemma 3.2 that if $f\in\mathcal{P}_0$, then it is sufficient that $a<1$. As mentioned above, for a distribution $R_1[a,b]$ we always have $a<1$, and thus such distributions have finite moments of all orders. Now let p be the distribution $R_1[a,b]$ and $h\in\mathcal{P}_+$ with $\mu_n(h)<\infty$. Then Theorem 3.3 gives that the moments of $p\vee h$ of order $j=1,2,\dots,n$ exist, are finite, and satisfy the recursion

$$\mu_j(p\vee h) = \frac{1}{1-a}\Sigma_{i=1}^{j} \binom{j-1}{i-1}\left[\binom{j}{i} a + b\right]\mu_i(h)\mu_{j-i}(p\vee h).$$

This recursion was derived by De Pril (1986).

4. Cumulants and the De Pril transform

4A. The De Pril transform φ_f of a function $f\in\mathcal{F}_0$ is defined by the recursion

$$\varphi_f(x) = \frac{1}{f(0)}[xf(x) - \Sigma_{y=1}^{x-1}\varphi_f(y)f(x-y)]. \qquad (x=0,1,\dots) \qquad (4.1)$$

By solving (4.1) with respect to $f(x)$ we obtain

$$f(x) = \frac{1}{x}\Sigma_{y=1}^{x}\varphi_f(y)f(x-y). \qquad (x=1,2,\dots) \qquad (4.2)$$

From (4.1) and (4.2) we see that the De Pril transform determines the function up to a multiplicative constant, that is, the set of all functions in $\mathcal{F}_0$ with De Pril transform φ_f is the set of functions cf where c is a positive constant. A distribution $f \in \mathcal{P}_0$ is uniquely determined by its De Pril transform by the scale condition $\mu_0(f)=1$.

From (4.2) we see that any function $f \in \mathcal{F}_0$ can be represented in the form $R_\infty[0,\mathbf{b}]$ with $b_x = \varphi_f(x)$ $(x=1,2,...)$

4B. The following theorem gives a relation between the cumulants and the De Pril transform of a function $f \in \mathcal{F}_0$.

Theorem 4.1. *Let n be a non-negative integer and* $f \in \mathcal{F}_0$ *with*

$$\Sigma_{x=1}^{\infty} x^{n-1} |\varphi_f(x)| < \infty. \tag{4.3}$$

Then the cumulants of f of order $j=0,1,...,n$ *exist and are finite and given by*

$$\kappa_j(f) = I(j=0) \ln f(0) + \Sigma_{x=1}^{\infty} x^{j-1} \varphi_f(x). \tag{4.4}$$

Proof. From Lemma 1 in Dhaene & De Pril (1994) we obtain

$$\mu_0(f) = f(0) \exp\left[\Sigma_{x=1}^{\infty} \frac{\varphi_f(x)}{x}\right]. \tag{4.5}$$

We see that $\mu_0(f)>0$, and thus our definition of cumulants can be applied. As $\kappa_0(f) = \ln \mu_0(f)$, we see that (4.4) holds for $j=0$.

From Theorem 3.2 follows that the moments of f of orders $1,2,..,r$ exist, are finite, and satisfy the recursion

$$\mu_j(f) = \Sigma_{i=1}^{j} \binom{j-1}{i-1} \mu_{i-1}(\varphi_f)\mu_{j-i}(f). \qquad (j=1,2,\ldots,n) \qquad (4.6)$$

As the moments are finite, the cumulants are also finite, and comparison of (4.6) with (2.6) gives that (4.4) is also satisfied for $j=1,2,\ldots,n$.

This completes the proof of Theorem 4.1. Q.E.D.

Dhaene & Sundt (1994) showed that the De Pril transform of a function f in the form $R_k[\mathbf{a},\mathbf{b}]$ satisfies the recursion

$$\varphi_f(x) = xa_x + b_x + \Sigma_{y=1}^{k} a_y\varphi_f(x-y). \qquad (x=1,2,\ldots) \qquad (4.7)$$

We shall now apply this recursion to obtain a sufficient condition for the condition (4.3) in Theorem 4.1 in the case when f is in the form $R_k[\mathbf{a},\mathbf{b}]$ with $k<\infty$.

Theorem 4.2. *If $f\in\mathcal{F}_0$ is in the form $R_k[\mathbf{a},\mathbf{b}]$ with $k<\infty$ and $\mu_0(|\mathbf{a}|)<1$, then $\mu_j(|\varphi_f|)<\infty$ for all j.*

Proof. For $g(0)$ sufficiently large, we can choose a function $g\in\mathcal{F}_0$ in the form $\mathrm{R}_k[|\mathbf{a}|,0]$ such that $|\varphi_f(x)|<g(x)$ for $x=1,2,\ldots,k$. Then it is easily shown by induction that this inequality is also satisfied for $x=k+1,k+2,\ldots$ This implies that $\mu_j(|\varphi_f|)\le\mu_j(g)$ for all positive integers j. By Lemma 3.1 $\mu_j(g)<\infty$ for all j, and thus the theorem is proved. Q.E.D.

We saw that the condition $\mu_0(|\mathbf{a}|)<1$ in Lemma 3.1 could be relaxed to $\mu_0(\mathbf{a}^+)<1$ if $f\in\mathcal{P}_0$. Such a modification of Theorem 4.2 is not possible. Let f be the distribution $R_1[a,b]$. From (4.7) we easily obtain

$$\varphi_f(x) = (a+b)a^{x-1}. \qquad (x=1,2,..) \qquad (4.8)$$

We see that for this distribution and any j $\mu_j(|\varphi_f|)<\infty$ if and only if $|a|<1$. For $a<-1$ $\mu_j(|\varphi_f|)=\infty$ for all j although the moments of all orders of this distribution exist. From (3.9) we see that in this case f is the binomial distribution (3.8) with $\pi>\frac{1}{2}$.

4C. In Theorem 3.2 we deduced a recursion for the moments of a function in the form $R_k[\mathbf{a},\mathbf{b}]$. Let us now apply Theorems 4.1–2 and the recursion (4.7) to deduce a similar recursion for the cumulants. For simplicity we restrict to the case $k<\infty$.

Theorem 4.3. *Let $f\in\mathcal{F}_0$ be in the form $R_k[\mathbf{a},\mathbf{b}]$ with $k<\infty$ and $\mu_0(|\mathbf{a}|)<1$. Then the cumulants of f of all orders exist, are finite, and satisfy the recursion*

$$\kappa_j(f) = \frac{1}{1-\mu_0(\mathbf{a})}\left[\mu_j(\mathbf{a}) + \mu_{j-1}(\mathbf{b}) + \Sigma_{i=1}^{j-1}\begin{bmatrix} j-1 \\ i \end{bmatrix}\mu_i(\mathbf{a})\kappa_{j-i}(f)\right]. \qquad (j=1,2,...) \qquad (4.9)$$

Proof. From Theorem 4.2 follows that the cumulants exist and are finite.

For $j=1,2,...$, we have

$$\kappa_j(f) = \Sigma_{x=1}^{\infty} x^{j-1}\varphi_f(x) = \Sigma_{x=1}^{\infty} x^{j-1}\left[xa_x + b_x + \Sigma_{y=1}^{k} a_y\varphi_f(x-y)\right] =$$
$$\mu_j(\mathbf{a}) + \mu_{j-1}(\mathbf{b}) + \Sigma_{y=1}^{k} a_y \Sigma_{x=1}^{\infty} \varphi_f(x)(x+y)^{j-1} =$$
$$\mu_j(\mathbf{a}) + \mu_{j-1}(\mathbf{b}) + \Sigma_{y=1}^{k} a_y \Sigma_{x=1}^{\infty} \varphi_f(x)\, \Sigma_{i=0}^{j-1}\begin{bmatrix} j-1 \\ i \end{bmatrix} y^i x^{j-i-1} =$$
$$\mu_j(\mathbf{a}) + \mu_{j-1}(\mathbf{b}) + \Sigma_{i=0}^{j-1}\begin{bmatrix} j-1 \\ i \end{bmatrix} \Sigma_{y=1}^{k} y^i a_y\, \Sigma_{x=1}^{\infty} x^{j-i-1}\varphi_f(x) =$$
$$\mu_j(\mathbf{a}) + \mu_{j-1}(\mathbf{b}) + \Sigma_{i=0}^{j-1}\begin{bmatrix} j-1 \\ i \end{bmatrix}\mu_i(\mathbf{a})\kappa_{j-i}(f),$$

from which we obtain (4.9). Q.E.D.

From (4.9) we obtain that if $\mathbf{a}$=0, then $\kappa_j(f)=\mu_{j-1}(\mathbf{b})$ (j=1,2,..). This result is obvious as in this case $b_x=\varphi_f(x)$ (x=1,2,...).

In particular we obtain from Theorem 4.3

$$\kappa_1(f) = \frac{\mu_1(\mathbf{a})+\mu_0(\mathbf{b})}{1-\mu_0(\mathbf{a})} \qquad \kappa_2(f) = \frac{\mu_2(\mathbf{a})+\mu_1(\mathbf{b})+\mu_1(\mathbf{a})\kappa_1(f)}{1-\mu_0(\mathbf{a})}.$$

For the case $f \in \mathcal{P}_0$, these expressions were given by Sundt (1992).

Analogous to the deduction of Theorem 3.3 we can deduce from Theorem 4.3 and Corollary 3.1 a recursion for the cumulants of a compound function with counting function in the form $R_k[\mathbf{a},\mathbf{b}]$ with $k<\infty$ and severity function in $\mathcal{F}_+$ with a finite support.

5. Approximations to distributions

5A. The De Pril transform can be a practical tool for evaluation of distributions in $\mathcal{P}_0$. We have already discussed some of its properties for distributions in the form $R_k[\mathbf{a},\mathbf{b}]$. More generally, the following two theorems were proved within the context of distributions by Sundt (1995) and extended to more general functions by Dhaene & Sundt (1994).

Theorem 5.1. *The convolution of a finite number of functions in $\mathcal{F}_0$ is a function in $\mathcal{F}_0$, and its De Pril transform is the sum of the De Pril transforms of these functions.*

Theorem 5.2. *If* $p \in \mathcal{F}_0$ *and* $h \in \mathcal{F}_+$, *then*

$$\varphi_{p \vee h}(x) = x\, \Sigma_{y=1}^{x} \frac{\varphi_p(y)}{y}\, h^{y*}(x).$$

Theorem 5.2 can also be obtained from Corollary 3.1 with **a**=**0** and $k=\infty$.

Unfortunately, although results like Theorems 5.1 and 5.2 may seem convenient, numerical evaluations may sometimes be rather time-consuming, and it is therefore of interest to study more computationally convenient approximations to De Pril transforms. Dhaene & Sundt (1994) discussed error bounds related to such approximations. Theorem 4.1 gives us another way to assess the quality of the approximation; we can compare the cumulants of the approximation with the cumulants of the exact distribution.

5B. We want to approximate $f \in \mathcal{P}_0$ by a function $f' \in \mathcal{F}_0$. We see that one way to reduce the time-consumption related to application of De Pril transforms, is to let $\varphi_{f'}(x)=0$ for x greater than some positive integer r.

For all positive integers r, let $\mathcal{F}_0^{(r)}$ denote the class of all functions $g \in \mathcal{F}_0$ for which $\varphi_g(x)=0$ for $x>r$. A function $g \in \mathcal{F}_0^{(r)}$ is uniquely determined by the r+1 quantities $g(0)$ and $\varphi_g(1),...,\varphi_g(r)$.

The condition (4.3) of Theorem 4.1 is obviously satisfied for $g \in \mathcal{F}_0^{(r)}$. Thus the cumulants and moments of g of all orders exist, and we have

$$\kappa_j(g) = I(j=0) \ln g(0) + \Sigma_{x=1}^{r} x^{j-1} \varphi_g(x). \qquad (j=0,1,....) \tag{5.1}$$

Dhaene & Sundt (1994) discuss some classes of approximations $f' \in \mathcal{F}_0^{(r)}$ to f. A simple and natural choice is to let $f'=f^{(r)}$ defined by

$$f^{(r)}(0) = f(0)$$

$$\varphi_{f^{(r)}}(x) = \varphi_f(x) I(x \leq r). \qquad (x=1,2,\ldots)$$

By considering (4.1) and (4.2) we see that the approximation $f^{(r)}$ can be interpreted as if we determine $f^{(r)}(0)$ and $\varphi_{f^{(r)}}(x)$ for $x=1,\ldots,r$ so that the approximation is exact for $f(x)$ for $x=0,1,\ldots,r$.

Considering Theorem 4.1, it seems natural to introduce another approximation $\mathfrak{f}^{(r)} \in \mathcal{F}_0^{(r)}$ where we instead of matching the probabilities up to r match the moments (or, equivalently, the cumulants) of orders $0,1,\ldots,r$, that is, $\kappa_j(f) = \kappa_j(\mathfrak{f}^{(r)})$ $(j=0,1,\ldots,r)$. Thus $\varphi_{\mathfrak{f}^{(r)}}$ is determined by the r linear equations

$$\Sigma_{x=1}^{r} x^{j-1} \varphi_{\mathfrak{f}^{(r)}}(x) = \kappa_j(f), \qquad (j=1,\ldots,r) \tag{5.2}$$

and as we should have $\mu_0(\mathfrak{f}^{(r)})=1$, we obtain from (4.5)

$$\mathfrak{f}^{(r)}(0) = \exp\left[-\Sigma_{x=1}^{r} \frac{\varphi_{\mathfrak{f}^{(r)}}(x)}{x}\right]. \tag{5.3}$$

5C. In this subsection we shall look at the special case when f is the Bernoulli distribution defined by

$$f(0) = 1-\pi \qquad f(1) = \pi. \qquad \left[0<\pi<\frac{1}{2}\right] \tag{5.4}$$

From (3.9) and (4.8) we obtain

$$\varphi_f(x) = -\left[\frac{\pi}{\pi-1}\right]^x. \qquad (x=1,2,\ldots) \tag{5.5}$$

Insertion of $\mu_j(f)=\pi$ (j=1,2,...) in (2.8) gives

$$\kappa_j(f) = \pi \left\{1 - \Sigma_{i=1}^{j-1} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix} \kappa_i(f)\right\}, \qquad (j=1,2,\ldots)$$

from which we easily see by induction that $\kappa_j(f)$ can be expressed as a polynomial in π of order j. The following theorem gives a closed form expression for $\kappa_j(f)$.

Theorem 5.3. *For any positive integer r, the cumulants of order j=1,2,...,r of the Bernoulli distribution f given by* (5.4) *are given by*

$$\kappa_j(f) = \Sigma_{x=1}^{r} x^{j-1}(-1)^{x+1} \Sigma_{y=x}^{r} \pi^y \begin{bmatrix} y-1 \\ x-1 \end{bmatrix}. \tag{5.6}$$

Proof. From (4.4) and (5.5) we obtain for j=1,2,...,r

$$\kappa_j(f) = \Sigma_{x=1}^{\infty} x^{j-1} \varphi_f(x) = \Sigma_{x=1}^{\infty} x^{j-1}(-1)^{x+1}\pi^x(1-\pi)^{-x} =$$
$$\Sigma_{x=1}^{\infty} x^{j-1}(-1)^{x+1}\pi^x \Sigma_{y=0}^{\infty} \begin{bmatrix} x+y-1 \\ y \end{bmatrix} \pi^y = \Sigma_{x=1}^{\infty} x^{j-1}(-1)^{x+1} \Sigma_{y=x}^{\infty} \begin{bmatrix} y-1 \\ x-1 \end{bmatrix} \pi^y,$$

which gives

$$\kappa_j(f) = \Sigma_{y=1}^{\infty} \pi^y \Sigma_{x=1}^{y} x^{j-1}(-1)^{x+1} \begin{bmatrix} y-1 \\ x-1 \end{bmatrix}. \tag{5.7}$$

We have earlier pointed out that $\kappa_j(f)$ can be expressed as a polynomial in π of order j. Thus, in (5.7) the coefficient of π^y must be equal to zero for all $y>r$, that is,

$$\kappa_j(f) = \Sigma_{y=1}^{r} \pi^y \Sigma_{x=1}^{y} x^{j-1}(-1)^{x+1} \begin{bmatrix} y-1 \\ x-1 \end{bmatrix},$$

from which we obtain (5.6) by interchanging the order of summation. Q.E.D.

By letting $r=j$ in (5.6) we obtain

$$\kappa_j(f) = \Sigma_{x=1}^{j} x^{j-1}(-1)^{x+1} \Sigma_{y=x}^{j} \pi^y \begin{bmatrix} y-1 \\ x-1 \end{bmatrix}. \qquad (j=1,2,\ldots)$$

We shall now consider the approximations $f^{(r)}$ and $\mathfrak{f}^{(r)}$ of f. For the former approximation, insertion of (5.5) in (5.1) gives

$$\kappa_j(f^{(r)}) = -\Sigma_{x=1}^{r} x^{j-1} \left[\frac{\pi}{\pi-1}\right]^x. \qquad (j=1,2,\ldots,r)$$

By comparing (5.2) and (5.6) we easily see that

$$\varphi_{\mathfrak{f}(r)}(x) = (-1)^{x+1} \Sigma_{y=x}^{r} \pi^y \begin{bmatrix} y-1 \\ x-1 \end{bmatrix} = (-1)^{x+1} x \Sigma_{y=x}^{r} \frac{\pi^y}{y} \begin{bmatrix} y \\ x \end{bmatrix}, \quad (x=1,2,\ldots,r) \qquad (5.8)$$

This gives

$$\Sigma_{x=1}^{r} \frac{\varphi_{\mathfrak{f}(r)}(x)}{x} = \Sigma_{x=1}^{r} (-1)^{x+1} \Sigma_{y=x}^{r} \frac{\pi^y}{y} \begin{bmatrix} y \\ x \end{bmatrix} = \Sigma_{y=1}^{r} \frac{\pi^y}{y} \Sigma_{x=1}^{y} (-1)^{x+1} \begin{bmatrix} y \\ x \end{bmatrix} = \Sigma_{y=1}^{r} \frac{\pi^y}{y},$$

and by insertion in (5.3) we obtain

$$\mathfrak{f}^{(r)}(0) = \exp\left[-\Sigma_{y=1}^{r} \frac{\pi^y}{y}\right]. \qquad (5.9)$$

5D. We shall now consider approximations to compound distributions by approximating the counting distribution and keeping the severity distribution unchanged.

Let $p \in \mathcal{P}_0$ and $h \in \mathcal{P}_+$. We want to approximate $p \vee h$ by $q \vee h$ with $q \in \mathcal{F}_0^{(r)}$. From Theorem 5.2 we obtain

$$\varphi_{q\vee h}(x) = x\,\Sigma_{y=1}^{r} \frac{\varphi_q(y)}{y} h^{y*}(x). \qquad (x=1,2,\ldots) \qquad (5.10)$$

This gives

$$\mu_{n-1}(|\varphi_{q\vee h}|) = \Sigma_{x=1}^{\infty} x^{n-1} |\varphi_{q\vee h}(x)| \le \Sigma_{x=1}^{\infty} x^n\,\Sigma_{y=1}^{r} \frac{|\varphi_q(y)|}{y} h^{y*}(x) = \Sigma_{y=1}^{r} \frac{|\varphi_q(y)|}{y} \mu_n(h^{y*}),$$

which is finite if the moments of h up to order n are finite. In that case we obtain from (5.10) and Theorem 4.1

$$\kappa_j(q\vee h) = \Sigma_{y=1}^{r} \frac{\varphi_q(y)}{y} \mu_j(h^{y*}). \qquad (j=1,2,\ldots,n) \qquad (5.11)$$

Furthermore, we have

$$\kappa_0(q\vee h) = \kappa_0(q) = \ln q(0) + \Sigma_{y=1}^{r} \frac{\varphi_q(y)}{y}. \qquad (5.12)$$

The moments and cumulants of $q \vee h$ can also be found from the moments and cumulants of q and h.

As the moments of orders $0,1,\ldots,r$ of the approximation $\tilde{p}^{(r)}$ to p are exact, we obtain that also the moments of orders $0,1,\ldots,r$ of the approximation $\tilde{p}^{(r)} \vee h$ to

$p \vee h$ are exact. Analogously we have that the probabilities up to r of the approximation $p^{(r)} \vee h$ to $p \vee h$ are exact.

5E. It is often convenient to interpret a distribution in $\mathcal{P}_0$ as a compound Bernoulli distribution with severity distribution in $\mathcal{P}_+$, that is, we represent the distribution $f \in \mathcal{P}_0$ by $p \vee h$ with

$$p(0) = 1-\pi = f(0) \qquad p(1) = \pi$$
$$h(x) = \frac{f(x)}{\pi} \qquad (x=1,2,\ldots)$$

We can approximate f by approximating p with a distribution in $\mathcal{F}_0^{(r)}$ and keeping h unchanged.

Let us first consider the approximation $p^{(r)}$. Insertion of (5.5) in (5.10)–(5.12) gives

$$\varphi_{p^{(r)} \vee h}(x) = -x\, \Sigma_{y=1}^{r} \frac{1}{y} \left[\frac{\pi}{\pi-1}\right]^{y} h^{y*}(x) \qquad (x=1,2,\ldots)$$
$$\kappa_j(p^{(r)} \vee h) = -\, \Sigma_{y=1}^{r} \frac{1}{y} \left[\frac{\pi}{\pi-1}\right]^{y} \mu_j(h^{y*}) \qquad (j=1,2,\ldots,n)$$
$$\kappa_0(p^{(r)} \vee h) = \kappa_0(p^{(r)}) = \ln(1-\pi) - \Sigma_{x=1}^{r} \frac{1}{y} \left[\frac{\pi}{\pi-1}\right]^{y}.$$

The approximation $p^{(r)} \vee h$ of f is the rth order De Pril approximation introduced by De Pril (1989).

We now turn to the approximation $\tilde{p}^{(r)}$. Insertion of (5.8) in (5.10) gives

$$\varphi_{\tilde{p}^{(r)} \vee h}(x) = x\, \Sigma_{y=1}^{r} (-1)^{y+1} h^{y*}(x)\, \Sigma_{z=y}^{r} \frac{\pi^z}{z} \begin{bmatrix} z \\ y \end{bmatrix}, \quad (x=1,2,\ldots)$$

and by rewriting this expression as

$$\varphi_{\tilde{p}^{(r)} \vee h}(x) = x\, \Sigma_{z=1}^{r}\, \Sigma_{y=1}^{z} \frac{(-1)^{y+1}}{z} \binom{z}{y} \pi^z h^{y*}(x) \quad (x=1,2,\ldots)$$

and comparing this formula and (5.9) with formula (54) in Dhaene & De Pril (1994), we see that the approximation $\tilde{p}^{(r)} \vee h$ of f is the rth order Hipp approximation introduced by Hipp (1986).

From our deductions follows in particular that the rth order Hipp approximation gives exact match of cumulants (and hence moments) of orders $0,1,\ldots,r$, when these cumulants exist. This property has also been shown by Dhaene, Sundt, & De Pril (1995). However, with our present deduction we have given a more extensive characterisation of the Hipp approximation. Like Hipp (1986) we represented the original distribution f as a compound distribution with Bernoulli counting distribution p and severity distribution h. Then, in our approximation we kept the severity distribution h unchanged, but approximated the counting distribution p with $\tilde{p}^{(r)}$, that is, the only approximation in $\mathcal{F}_0^{(r)}$ that gives exact match for the moments of orders $0,1,\ldots,r$.

We can give an analogous characterisation of the rth order De Pril approximation. The difference is that there we approximate the counting distribution p with $p^{(r)}$, that is, the only approximation in $\mathcal{F}_0^{(r)}$ that gives exact match for the probabilities up to r.

These characterisations show that the Hipp approximation is related to matching of moments whereas the De Pril approximation is related to matching of probabilities. This may indicate that the approximations may be appropriate in different situations; the De Pril transform when we are primarily interested in the approximated probabilities, the Hipp approximation when the approximated moments are more important.

As indicated in subsection 5D, we can apply the same principles when approximating compound distributions with other counting distributions.

5F. In the previous subsection we deduced the approximations of Hipp and De Pril as approximations of one distribution in $\mathcal{P}_0$. These approximations are usually presented in the more general framework of approximating a convolution of distributions in $\mathcal{P}_0$. The convolution is appproximated by approximating each of the distributions by the rth order Hipp resp. De Pril approximation. For further discussions and comparisons between these two classes of approximations we refer to De Pril (1989) and Dhaene & De Pril (1994).

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