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SOLUTION OF THE FOKKER-PLANCK EQUATION WITH BOUNDARY CONDITIONS BY FEYNMAN-KAC INTEGRATION

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Solution of the Fokker-Planck Equation with Boundary Conditions by Feynman-Kac Integration

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Abstract

In this paper, we apply the results about δ and δ'-function perturbations in order to formulate within the Feynman-Kac integration the solution of the forward Fokker-Planck equation subject to Dirichlet or Neumann boundary conditions. We introduce the concept of convex order to derive upper and lower bounds for path integrals with δ and δ'-functions in the integrand. We suggest the use of those bounds as an approximation for the solution.

Keywords: Feynman-Kac integration, Perturbations theory, SDE.

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1 Introduction

Diffusion processes can be related to quantum mechanical particles with convenient Lagrangians through the Feynman *sum over all paths* formalism. As proved in a previous paper [1], the transition probability of the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

can be expressed as a path integral multiplied by a suitable gauge transform and the Schrödinger equation turns into the forward Fokker-Planck equation

$$\partial_t p = \frac{1}{2}\partial_x^2(\sigma^2 p) - \partial_x(\mu p) \tag{2}$$

with the standard conditions

$$\lim_{x\to+\infty} p(x,t) = 0 \quad \forall t, \qquad \lim_{x\to-\infty} p(x,t) = 0 \quad \forall t, \qquad p(x,0) = \delta(x - x_0)$$

where $p(x,t)$ denotes the probability for the particle to be located in x at time t starting in x_0.

In many applications, the boundary conditions at infinity appear like a very idealization and one may be willing to incorporate in $x = a$ an impermeable barrier to the potential and specify the behaviour of the particle while attaining this limit. Dirichlet and Neumann problems have been treated in the Feynman formalism using infinitely repulsive δ respectively δ' perturbations as done in [2] and [3]. In this paper, we apply those results to obtain closed expressions for the transition probabilities of a stochastic differential equation subject to an absorbing or a reflecting boundary in $x = a$. We use the inequality of Jensen together with the concept of comonoticity to derive analytical bounds.

The paper is organized as follows. In the first section, we recall the main results about δ and δ' perturbations. In section 2, we present expressions for the transition probabilities with absorbing and reflecting boundary by means of path integrals. In section 3, we extend the approximation derived in [1] to the present situation. In the last section,we illustrate the accuracy of the bounds on a numerical example. All the proofs are contained in the appendix together with some exact calculations.

2 δ and δ' Perturbations

The Green function for the one dimensional δ-function potential can be solved as done in [2] by summing all the terms of the perturbation expansion. Let $I_V(x_0, x, t)$ be the following Feynman integral

$$I_V(x_0, x, t) = \int_{(0,x_0)}^{(t,x)} D_{x_s} e^{-\frac{1}{2}\int_0^t (\frac{dx}{ds})^2 ds - \int_0^t V(x_s) ds} \tag{3}$$

solution of the Schrödinger equation with the standard conditions

$$\partial_t I_V(x_0, x, t) = \frac{1}{2}\partial_x^2 I_V(x_0, x, t) - I_V(x_0, x, t)V(x) \tag{4}$$

$$\lim_{x \to +\infty} I_V(x_0, x, t) = 0, \quad \lim_{x \to -\infty} I_V(x_0, x, t) = 0, \quad I_V(x_0, x, 0) = \delta(x - x_0).$$

Or equivalently in the Laplace domain,

$$sG_V(x_0, x, s) = \frac{1}{2}\partial_x^2 G_V(x_0, x, s) - G_V(x_0, x, s)V(x) + \delta(x - x_0) \tag{5}$$

with $\lim_{x \to \pm\infty} G_V(x_0, x, t) = 0$ where $G_V(x_0, x, s)$ denotes the Green function of $I_V(x_0, x, t)$ and reads

$$G_V(x_0, x, s) = \int_0^\infty e^{-st} I_V(x_0, x, t) dt. \tag{6}$$

As shown in [2], the Green function of the path integral $I_{V+\beta\delta(x-a)}$ can be expressed using (6),

$$G_{V+\beta\delta(x-a)}(x_0, x, s) = G_V(x_0, x, s) - \frac{G_V(x_0, a, s)G_V(a, x, s)}{G_V(a, a, s) - 1/\beta}. \tag{7}$$

It is straighforward to check that when β goes to infinity, $G_{V+\beta\delta(x-a)}(x_0, a, s)$ converges to zero. Hence, we get at the limit $G_V^D(x_0, x, s)$, the Dirichlet solution of the corresponding Schrödinger equation (5) in the Laplace domain. For the free particle, e.g. $V \equiv 0$, we omit V in the notations. More details about δ-function perturbation can be found in [2].

The δ'-function perturbation expansion leads to a summation of strongly intercorrelated terms without straightforward solution. Grosche [3] proposes a specific regularization procedure and obtains the following expression for the Green function of the path integral $I_{V+\beta\delta'(x-a)}$

$$G_{V+\beta\delta'(x-a)}(x_0,x,s) = G_V(x_0,x,s) - \frac{\partial_x G_V(x_0,a,s)\partial_{x_0}G_V(a,x,s)}{\partial_x\partial_{x_0}G_V(a,a,s)+1/\beta}. \tag{8}$$

It is straighforward to check that when β goes to infinity, $\partial_x G_{V+\beta\delta(x-a)}(x_0,a,s)$ converges to zero. Hence, we get at the limit $G_V^N(x_0,x,s)$, the Neumann solution of the corresponding Schrödinger equation (5) in the Laplace domain. Note that even in the simpliest case $V\equiv 0$, $G_{\beta\delta'(x-a)}(x_0,x,s)$ exibits a discontinuity in $x=a$ as $G(x_0,x,s)=\frac{1}{\sqrt{2s}}e^{-|x-x_0|\sqrt{2s}}$. So, we should keep in mind that the Green function $G_{\delta(x-a_-)+\beta\delta'(x-a)}$ is a priori different than $G_{\delta(x-a_+)+\beta\delta'(x-a)}$. More details about δ'-function perturbation can be found in [3].

3 Transition Probabilities with Boundary Conditions

In this section, we show how use can be made of δ and δ' perturbations to incorporate boundary conditions into the closed forms obtained in [1] for the transition probabilities of a stochastic differential equation. In order to avoid cumberstome notations, we present the results for stochastic differential equation with unit volatility but the method can be easily generalized to the more general case.

3.1 Absorbing boundary

Theorem 1 *Consider the following stochastic differential equation*

$$dX_t = \mu(X_t)dt + dW_t$$

where W_t is a standard Brownian motion. Assume that the diffusion process $X=\{X_s, s\in[0,t]\}$ is subject to an absorbing boundary at an attainable point a. We define the path integral $I_\beta(x_0,x,t)$ as

$$I_\beta(x_0,x,t) = \int_{(0,x_0)}^{(t,x)} D_{x_s} e^{-\frac{1}{2}\int_0^t\left(\frac{dx}{ds}\right)^2 ds - \frac{1}{2}\int_0^t \mu^2(x)ds - \frac{1}{2}\int_0^t \partial_x\mu(x)ds - \beta\int_0^t \delta(x-a)ds}.$$

The transition probability of the stochastic process X_t starting in $x_0 > a$ can be written by means of $I_\beta(x_0, x, t)$

$$p(x_0, x, t) = \begin{cases} \lim_{\beta\to\infty} e^{\int_{x_0}^{x}\mu(z)dz} I_\beta(x_0, x_t, t), & x > a \\ \left[1 - \int_{a_+}^{+\infty} p(x_0, x, t)dx\right]\delta(x-a) & x = a \\ 0 & x < a, \end{cases}$$

and $p(x_0, x, t)$ satisfies the forward Fokker-Planck equation

$$\partial_t p = \frac{1}{2}\partial_x^2 p - \partial_x(\mu p)$$

with the conditions

$$\lim_{x\to a_+} p(x_0, x, t) = 0 \quad \forall t, \qquad \lim_{x\to\infty} p(x_0, x, t) = 0 \quad \forall t, \qquad p(x_0, x, 0) = \delta(x - x_0)$$

The proof of this theorem is provided in appendix A.

Remark 1 *In the more general case $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$, one may apply the Ito lemma to $\phi(X_t)$ where*

$$\phi(x) = \int^x \frac{dz}{\sigma(z)}$$

to normalize the volality and impose an absorbing boundary in $\phi(a)$.

Remark 2 *Of course, theorem 1 can be adapted when $x_0 < a$.*

3.2 Reflecting boundary

The reflecting boundary generates a an additional term in the drift. The reflected diffusion starting in $x_0 > a$ is the solution of the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \frac{1}{2}L_t^{a+}(X) \tag{9}$$

where $L^{a+}(X)$ is the right-local time associated with the semimartingal X and can be bravely defined for our purpose as

$$L_t^{a+}(X) = \int_0^t \delta(X_s - a_+)ds. \tag{10}$$

A rigourous approach to stochastic differential equation involving their local time can be found in Legall [5].

Theorem 2 *Consider the following stochastic differential equation*

$$dX_t = \mu(X_t)dt + dW_t + \frac{1}{2}L_t^{a+}(X)$$

where W_t is a standard Brownian motion. We define the path integral $I_\beta(x_0, x, t)$ as

$$I_\beta(x_0, x, t) = \int_{(0,x_0)}^{(t,x)} D_{x_s} e^{-\frac{1}{2}\int_0^t \left(\frac{dx}{ds}\right)^2 ds - \frac{1}{2}\int_0^t \mu(x)ds - \frac{1}{2}\int_0^t \partial_x \mu^2(x)ds - \frac{\mu(a)}{2}\int_0^t \delta(x-a_+)ds - \beta\int_0^t \delta'(x-a)ds}$$

The transition probability of the stochastic process X_t starting in $x_0 > a$ can be written by means of $I_\beta(x_0, x, t)$

$$p(x_0, x, t) = \begin{cases} \lim_{\beta\to\infty} e^{\int_{x_0}^x \mu(z)dz} I_\beta(x_0, x_t, t), & x \geq a \\ 0 & x < a, \end{cases}$$

and $p(x_0, x, t)$ satisfies the forward Fokker-Planck equation

$$\partial_t p = \frac{1}{2}\partial_x^2 p - \partial_x(\mu p)$$

with the conditions

$$\lim_{x\to a_+} \frac{1}{2}\partial_x p(x_0, x, t) = \mu(x)p(x_0, x, t), \qquad \lim_{x\to\infty} p(x_0, x, t) = 0 \quad \forall t,$$

$$p(x_0, x, 0) = \delta(x - x_0).$$

The proof of this theorem is provided in appendix A.

Remark 3 *In the more general case $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \frac{1}{2}L_t^{a+}(X)$, one may apply the generalized Tanaka formula to $Y_t = \phi(X_t)$ where*

$$\phi(x) = \int^x \frac{dz}{\sigma(z)}$$

to normalize the volatility. We can check using properties of local time that the process Y_t is reflected in $\phi(a)$.

Remark 4 *Since δ'-function perturbation leads to propagator discontinuous in a, the right-delta potentiel $\delta(x - a_+)$ defines in an unambiguous way the path integral $I_\beta(x_0, x, s)$. This is closely related with the discontinuity of the symmetric local time for reflected diffusion.*

Remark 5 *This is a nice illustration in the path integral formalism of the Girsanov transformation for reflected diffusion proposed by Kinkladze [6]*

Remark 6 *Of course, theorem 2 can be adapted when $x_0 < a$.*

4 Bounds for Path Integrals With δ and δ'-Functions in the Integrand

In the previous section, we derived closed forms for the solution of the forward Fokker-Planck equation with an absorbing or a reflecting boundary. For the computations we can rely on exact calculations from quantum mechanic but in most cases, the functional integrals remain unsolved. Nevertheless, we may derive an upper and a lower bound. A first step is to express the Feynman integral as an expectation over all paths with respect to an convenient measure.

In order to make things clear, we use the notation

$$p(x_0, x, t) = C(x_0, x, t) I_V(x_0, x, t) \tag{11}$$

where $I_V(x_0, x, t)$ is the general path integral

$$I_V(x_0, x, t) = \int_{(0,x_0)}^{(t,x)} D_{x_s} e^{-\frac{1}{2}\int_0^t \left(\frac{dx}{ds}\right)^2 ds - \int_0^t V(x_s) ds} \tag{12}$$

where the potential may include δ or(/and) δ'-function The trick is to decompose the potential $V(x)$ as a sum $V_1(x) + V_2(x)$ such that the path integral corresponding to V_2 is tractable. The following lemma will permit us to derive analytical bounds making use of the properties of convex ordered risks on the hand and of the Jensen inequality on the other.

Lemma 1 *The transition probabilities $p(x_0, x, t)$ defined as in (11) can be written for any decomposition $V(x) = V_1(x) + V_2(x)$ as*

$$p(x_0, x, t) = C(x_0, x, t) I_{V_2}(x_0, x, t) E_X \left[e^{-\int_0^t V_1(X_s) ds} \right]$$

where the expectation is taken over all paths starting in x_0 with fixed final point x and the following marginal distributions $F(x_s)$,

$$\frac{d}{dx_s} Prob\,(X_s \le x_s) = \frac{d}{dx_s} F(x_s) = \frac{I_{V_2}(x_0, x_s, s) I_{V_2}(x_s, x, t)}{I_{V_2}(x_0, x, t)}.$$

The proof of this lemma is straightforward starting from the path integral representation of $p(x_0, x, t)$. The method is based on a paper of the same authors [1]. Nevertheless, the most relevant concepts are briefly recalled.

4.1 Upper bound

The method we propose to find accurate upper bound for the transition probability makes use of convex order. Let's recall the necessary results about convex ordering of random variables.

The variable A is said to be smaller than B in convex ordering,

$$A \leq_{cx} B \tag{13}$$

if for each convex function $u : \mathbb{R} \rightarrow \mathbb{R} : x \rightarrow u(x)$ the expected values (provided they exist) are ordered as

$$E[u(A)] \leq E[u(B)] \tag{14}$$

As a consequence, $E[A] = E[B]$ and

$$E[(A-k)_+] \leq E[(B-k)_+] \qquad \forall k, \tag{15}$$

with $(x)_+ = max(0, x)$.

If an expression is known for the stop-loss premium $E[(B-k)_+]$, the distribution of the variable B can be found as

$$Prob[B \leq k] = 1 + \frac{d}{dk} E[(B-k)_+]. \tag{16}$$

The notion of convex ordering can be extended from two single variables to two sums of variables, discrete or continuous. The results are summarized in the following two propositions. If $F_X(x) = Prob[X \leq x]$ denotes a distribution, we make use of the notation hereafter for the inverse distribution

$$F_X^{-1}(p) = inf\{x \in \mathbb{R} : F_X(x) \geq p\}. \tag{17}$$

Lemma 2 *Consider a sum of functions of random variables*

$$A = g_1(X_1) + g_2(X_2) + \ldots + g_n(X_n)$$

and an arbitrary random variable U that is uniformly distributed on $[0, 1]$, define the related stochastic quantity

$$B = F^{-1}_{g_1(X_1)}(U) + F^{-1}_{g_2(X_2)}(U) + \ldots + F^{-1}_{g_n(X_n)}(U).$$

Then

$$A \leq_{cx} B.$$

Lemma 3 *Consider a functional integration*

$$A = \int_0^t g(X_s)ds,$$

and for an arbitrary random variable U that is uniformly distributed on $[0,1]$*, define the related stochastic quantity*

$$B = \int_0^t F^{-1}_{g(X_s)}(U)ds.$$

Then

$$A \leq_{cx} B.$$

An application of the concept of convex upperbounds to path integrals brings us to the following result:

Theorem 3 *For a path integral with structure as mentioned in equation (12), an upperbound can be found as*

$$I_V(x_0, x, t) \leq I_V^{upp}(x_0, x, t)$$

with

$$I_V^{upp}(x_0, x, t) = I_{V_2}(x_0, x, t) E_U \left[e^{-\int_0^t F^{-1}_{V_1(X_s)}(U)ds} \right]$$

where U is uniformly distributed on $[0,1]$ *and* $F_X(x) = F(x)$ *is defined in lemma 1.*

4.2 Lower bound

In order to improve our upperbound, we also need a accurate lowerbound. The present result mainly originates from an application of the famous inequality of *Jensen* and was proposed by Feynman [7].

Theorem 4 *For a path integral with structure as mentioned in equation (12), a lowerbound can be found as*

$$I_V(x_0, x, t) \geq I_V^{low}(x_0, x, t)$$

with

$$I_V^{low}(x_0, x, t) = I_{V_2}(x_0, x, t) E_{X_\tau} \left[e^{-\int_0^\tau E_X[V_1(X_s)]ds} e^{-\int_\tau^t E_X[V_1(X_s)]ds} | X_\tau = x_\tau \right]$$

where E_X *means an expectation over all paths with fixed starting and final points and marginal distributions* $F(x_s)$ $as\ \ defined\ \ in\ \ lemma 1.$

4.3 Application for transition probabilities with boundary conditions

The previous inequalities can be used in order to approximate path integrals and the norm of the difference between the bounds can be interpreted as an indicator for the accuracy. By the way, the functional integral is reduced to two single variable integrals. The following lemmae propose handly decompositions $V(x) = V_1(x) + V_2(x)$ in order to estimate the transition probabilities of diffusions subject to a an absorbing or a reflecting boundary.

4.3.1 Absorbing Boundary

Lemma 4 *Consider the transition probability derived in theorem 1, we have that*

$$C(x_0, x, t) I_V^{low}(x_0, x, t) \leq p(x_0, x, t) \leq C(x_0, x, t) I_V^{upp}(x_0, x, t)$$

where $V(x) = V_1(x) + V_2(x)$ *with*

$$\begin{aligned} V_1(x) &= \frac{1}{2}[\mu(x)^2 + \partial_x \mu(x)] \\ V_2(x) &= \lim_{\beta \to \infty} \beta \delta(x - a). \end{aligned}$$

This lemma is a straightforward application of lemma 1, theorem 3 and 4 and one may check that the path integral corresponding to $V_2(x)$ is the density function of the absorbed Brownian motion.

4.3.2 Reflecting Boundary

Lemma 5 *Consider the transition probability derived in theorem 2, we have that*

$$C(x_0, x, t) I_V^{low}(x_0, x, t) \leq p(x_0, x, t) \leq C(x_0, x, t) I_V^{upp}(x_0, x, t)$$

where $V(x) = V_1(x) + V_2(x)$ *with*

$$\begin{aligned} V_1(x) &= \frac{1}{2}[\mu(x)^2 + \partial_x \mu(x)] \\ V_2(x) &= \frac{\mu(a)}{2} \delta(x - a_+) + \lim_{\beta \to \infty} \beta \delta'(x - a). \end{aligned}$$

This lemma is a straightforward application of lemma 1, theorem 3 and 4. The path integral together with the Green function corresponding to $V_2(x)$ are provided in appendix B.

Remark 7 *The bounds of lemma 5 are no longer density function of reflected diffusions as the total mass is smaller respectively larger than one. To restore this desirable property, we suggest the use of the following convex combination*

$$\tilde{p}(x_0, x, t) = C(x_0, x, t) \left[z I_V^{low}(x_0, x, t) + (1 - z) I_V^{upp}(x_0, x, t) \right]$$

where z is choosen such that $\int_a^{+\infty} \tilde{p}(x_0, x, t) dx = 1$.

5 Numerical Illustration

Finally, we illustrate the accuracy of the bounds on the following bimodal stochastic differential equation[1]

$$dX_t = \left(X_t - X_t^3 \right) dt + dW_t. \tag{18}$$

The domain of the process X_t is $[-\infty, +\infty]$. We impose sucessively an absorbing and a reflecting boundary[2] in $x = 0$. No transformation is needed as the volatility is equal to one. As $\mu(0) = 0$, we avoid straightforward complications. Note that the bimodal property of the stationnary process X_t makes the approximation of the transition probabilities uneasy.

Figure 1 shows the bounds with an absorbing boundary. A simple integration provides us with an interval for the probability of absorbtion,

$$0.536 \leq 1 - Prob\left(inf\{X_s | s \in [0, t], X_0 = x_0\} > 0 \right) \leq 0.602$$

In figure 2, we plot the bounds together with the convex combination in case of reflecting boundary. The lower bound seems to perform beter but the upper bound remains interesting.

[1]This SDE originates in finance to model spot rate and was first proposed by Y. Ait-Sahalia [9]

[2]Forcing the spot rate to stay above zero circumvents the main drawback of the model

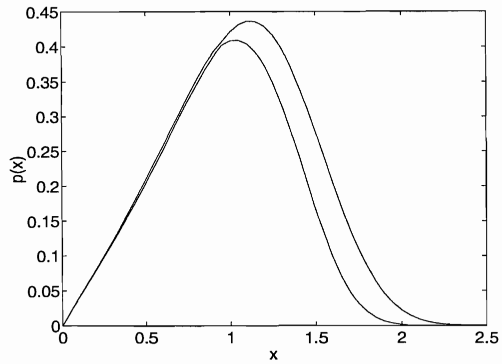

Figure 1: Bounds for the transition probability with absorbtion in $x = 0$, for $t = 1$ and starting point $x_0 = 0$

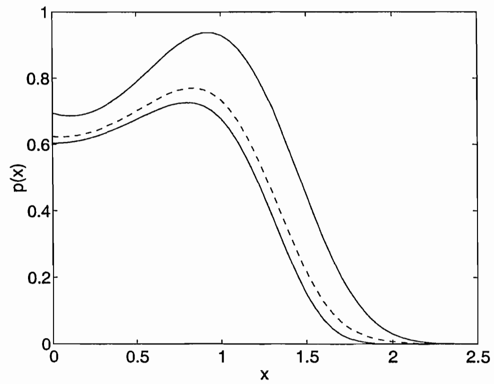

Figure 2: Approximation of the transition probability with reflection in $x = 0$, for $t = 1$ and starting point $x_0 = 0$

6 Conclusion

Through the Feynman path integrals formalism, we derived bounds for the solution of the Fokker-Planck equation subject to various boundary conditions. Despite the inversion of $F(x)$, the bounds can be easily computed. Applications are numerous an will be the subject of further research as well as the optimization of the upperbound.

A Proof of the Results

A.1 Proof of theorem 1

We start to prove that $p(x_0, x, t)$ is solution of the forward Fokker-Planck equation. We denote by $G_V^D(x_0, x, s)$ the limit of the Green function of the path integral $I_\beta(x_0, x, t)$, β going to infinity. From δ-function perturbation theory (7), we deduce that

$$\frac{1}{2}\partial_x^2 G_V^D(x_0, x, s) = sG_V^D(x_0, x, s) + G_V^D(x_0, x, s)V(x) - \delta(x - x_0)$$

with the boundary condition $G_V^D(x_0, a, s) = 0$ where $V(x) \equiv \frac{1}{2}[\mu(x)^2 + \partial_x \mu(x)]$. The desired result follows after multiplication by $e^{\int_{x_0}^{x} \mu(z)dz}$ and inversion of the Laplace transform.

Hence, from probability theory, we know that $p(x_0, x, t)$ is the transition probability of the absorbed process X_t, see [10].

A.2 Proof of theorem 2

We start to prove that $p(x_0, x, t)$ is solution of the forward Fokker-Planck equation. From δ'-function perturbation theory (8), the Green function $G_V^N(x_0, x, s)$ of the path integral

$$\lim_{\beta \to +\infty} \int_{(0,x_0)}^{(t,x)} D_{x_s} e^{-\frac{1}{2}\int_0^t \left(\frac{dx}{ds}\right)^2 ds - \int_0^t V(x)ds - \beta \int_0^t \delta'(x-a)ds}$$

where $V(x) \equiv \frac{1}{2}[\mu(x)^2 + \partial_x \mu(x)]$ satisfies

$$\frac{1}{2}\partial_x^2 G_V^D(x_0, x, s) = sG_V^D(x_0, x, s) + G_V^D(x_0, x, s)V(x) - \delta(x - x_0)$$

with the boundary condition $\partial_x G_V^N(x_0, a_+, s) = 0$. we denote by $\tilde{G}(x_0, x, s)$ the limit of the Green function of the path integral $I_\beta(x_0, x, t)$, β going to infinity. From δ-function perturbation theory (7),

$$\begin{aligned} \tilde{G}(x_0, x, s) &= G_V^N(x_0, x, s) - \frac{G_V^N(a_+, x, s)G_V^N(x_0, a_+, s)}{G_V^N(a_+, a_+, s) + 2/\mu(a)} \\ &= G_V^N(x_0, x, s) - \frac{\mu(a)}{2}\tilde{G}(x_0, a_+, s)G_V^N(a_+, x, s). \end{aligned}$$

As $\partial_x G_V^N(x_0, a_+, s) = 0$ except when $x_0 = a_+$ then $\partial_x G_V^N(x_0, a_+, s) = -2$, we conclude that $\partial_x \tilde{G}(x_0, a_+, s) = \mu(a)\tilde{G}(x_0, a_+, s)$. The desired result follows after multiplication of $\tilde{G}(x_0, a_+, s)$ by $e^{\int_{x_0}^{x} \mu(z)dz}$ and inversion of the Laplace transform.

Hence, from probability theory, we know that $p(x_0, x, t)$ is the transition probability of the reflected process X_t, see [10].

A.3 Proof of theorem 3

The path integral $I_V(x_0, x, t)$ can be written as

$$I_V(x_0, x, t) = I_{V_2}(x_0, x, t) E_X \left[e^{-\int_0^t V_1(X_s)ds} \right]$$

Applying lemma 3, we know that the variable $A = \int_0^t V_1(X_s)ds$ is smaller than $B = \int_0^t F^{-1}_{V_1(X_s)}(U)ds$ in convex ordering. Since the exponential function is convex, it follows immediately from the definition of convex ordering that

$$I_V(x_0, x, t) \le I_V^{upp}(x_0, x, t).$$

A.4 Proof of theorem 4

We start by writing the path integral $I_V(x_0, x, t)$ as

$$I_V(x_0, x, t) = I_{V_2}(x_0, x, t) E_\Lambda \left[E_X \left\{ e^{-\int_0^t V(X_\tau)d\tau} \right\} |\Lambda \right]$$

for an arbitrary random variable Λ. The result follows as an application of Jensen inequality with $\Lambda = X_s$.

B Exact Calculations

$$\begin{aligned} G_{V_2}(x_0, x, s) &= \int_0^\infty e^{-st} \int_{(0,x_0)}^{(t,x)} D_{x_s} e^{-\frac{1}{2}\int_0^t \left(\frac{dx}{ds}\right)^2 ds - \frac{\mu(a)}{2}\int_0^t \delta(x-a_+)ds - \beta \int_0^t \delta'(x-a)ds} \\ &= G^N(x_0, x, s) - \frac{G^N(a_+, x, s) G^N(x_0, a_+, s)}{G^N(a_+, a_+, s) + 2/\mu(a)} \end{aligned}$$

Inserting the expression of $G^N(x_0, x, s)$, we obtain

$$G_{V_2}(x_0, x, s) = \begin{cases} G^N(x_0, x, s) + 2\mu(a)\frac{e^{-\sqrt{2s}(|x-a|+|a-x_0|)}}{\sqrt{2s}(\sqrt{2s}-\gamma)}, & x \geq a \\ 0 & x < a. \end{cases}$$

The inversion of the Laplace transform can be performed as done in [8]

$$I_{V_2}(x_0, x, t) = \begin{cases} I^N(x_0, x, t) + \mu(a)e^{\mu(a)k}e^{\frac{1}{2}\mu(a)^2 t}erfc\left(\mu(a)\sqrt{t/2} + \frac{k}{\sqrt{2t}}\right), & x_t \geq a \\ 0 & x_t < a. \end{cases}$$

with $k = (|x-a| + |a-x_0|)$ and $I^N(x_0, x, t)$ is the transition probability of the reflected Brownian motion.

References

[1] M.J. Goovaerts, A. De Schepper, M. Decamps*Transition Probabilities for Diffusion Equations by Means of Path Integral*, 2002, submitted.

[2] M.J. Goovaerts, A. Babcenco and J.T. Devreese, *A New Expansion Method in the Feynman Path Integral Formalism: Application to the One-Dimensional Delta-Function Potential*,J.Math.Phys,14 (1973) 554.

[3] C. Grosche , *$\delta'-$function Perturbations and Neumann Boundary Conditions by Path Integration*, J. Phys. A: Math gen., 28 (1995) L99-L105.

[4] M. Kac , *On Some Connections between Probability Theory and Differential and Integral Equations*, Proc. Second Symp. on Math. Stat. and Prob., Berkeley (1951), pp189-215.

[5] J.F. Legall, *Temps Locaux et équations différentielles stochastiques*, Lectures Notes in Math. 986, Springer-Verlag, NY, Berlin, 1982 .

[6] G.N.Kinkladze, *A Note on the Structure of Processes the Measure of Which is Abslotely Continuous with Respect to the Wiener Modulus Measure*, Stochastics, (1982) Vol. 8 pp39-44.

[7] R.P. Feynman, Hibbs A.R. , *Quantum Mechanics and Path Integrals*, McGraw-Hill Book Company, NY (1965), 365p.

[8] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover NY (1970).

[9] Y. Ait-Sahalia,*Transition Probabilities for Interest Rate and Other Nonlinear Diffusions*, The Journal of Finance, vol.LIV 4, 1999, pp1361-1395.

[10] Gikhman and Skorokhod, *Introduction to the Theory of Random Process*, Saunders Mathematics Books, 1963, chap 3.